

MULTISCALE ANALYSIS OF SIMILARITIES BETWEEN IMAGES ON RIEMANNIAN MANIFOLDS*

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Abstract. In this paper we study the problem of comparing two patches of an image defined on a Riemannian manifold, which can be defined by the image domain with a suitable metric depending on the image. The size of the patch will not be determined a priori, and we identify it with a variable scale. Our approach can be considered as a nonlocal extension (comparing two points) of the multiscale analyses defined using the axiomatic approach by Álvarez et al. [*Arch. Ration. Mech. Anal.*, 123 (1993), pp. 199–257]. Following this axiomatic approach, we can define a set of similarity measures that appear as solutions of a degenerate partial differential equation. This equation can be further specified in the linear case, and we observe that it contains as a particular instance the case of using weighted Euclidean distances as comparison measures. Finally, we discuss the case of some morphological scale spaces that exhibit a higher complexity.

Key words. multiscale analysis, similarity measures, degenerate parabolic equations

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1. Introduction. Our purpose in this paper is to compare two patches of an image defined on a Riemannian manifold, which can be defined by the image domain with a suitable metric depending on the image. The size of the patch will not be determined a priori, and we identify it with a variable scale. Our approach can be considered as a nonlocal extension (comparing two points) of the multiscale analyses defined using the axiomatic approach in [1].

Let us review the fundamentals of that approach. A multiscale analysis represents a given image at different scales of smoothing, the scale being related to the size of the neighborhood which is used to give an estimate of the brightness of the picture at a given point. It is a basic preprocessing step for shape recognition [27] (see [16, 6] and references therein).

The systematic study of multiscale analyses for images was the purpose of the axiomatic approach proposed in [1]. Based on a series of axioms which define the structure of the multiscale space and a set of geometric and photometric invariants, multiscale analyses were defined in terms of (viscosity) solutions of a parabolic equation. In the case of linear multiscale analysis they obtained the Gaussian scale space (already proposed and studied in [28, 24, 25, 49, 18, 17, 26, 48], using also an axiomatic approach in some of those papers). In addition to the Gaussian scale space, classification covers many of the classical models that were proposed in the literature, such as the Perona–Malik equation [34] (see also [9]), the Rudin–Osher–Fatemi model

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[36], and the mean curvature motion as proposed in [2].

Assuming the invariance under contrast changes (i.e., monotone rearrangements of the gray levels), multiscale analyses were given in terms of geometric equations [12, 11, 33] that diffuse the level sets of the image with functions of their principal curvatures. Following [1, 16], we refer to them as morphological scale spaces since they are related to a PDE formulation of mathematical morphology [39]. The case $N = 2$ is of particular interest and leads to the motion of level lines by a function of curvature [31, 13, 20, 21]. Of particular interest is the affine morphological scale space (AMSS) [1, 37, 38, 32], which is affine invariant and corresponds to motion of level lines by the power $1/3$ of its curvature. The case of scale spaces for three-dimensional (3D) images gives rise to geometric motions that depend on functions of the two principal curvatures of the level surfaces [1, 43] (e.g., mean and Gaussian curvatures). In the case of video sequences, the Galilean invariant scale spaces were characterized in a similar way to the 3D case, and Gaussian curvature was replaced by acceleration [1, 16, 14, 15].

Besides this unified trend, scale spaces based on anisotropic diffusion have been the object of systematic study by Weickert [46, 45], and, although they fall into the general set of nonlinear models described in [1], they were not axiomatically studied there. Finally, variational models also give a different approach to image diffusion. They are also basic ingredients in the regularization of inverse problems. Let us mention here the work of Rudin, Osher, and Fatemi [36], who introduced total variation as an image regularizer due to its ability to restore edges. A more general formulation is given in [23, 41, 22, 40], where the authors consider images defined on Riemannian manifolds where the metric depends on the image and reflects the anisotropy of the underlying problem (for edge preservation, for color image restoration, for texture analysis, etc.). Their basic energy functional is the Polyakov action, which is the extension of the Dirichlet integral to maps between Riemannian manifolds [23, 41].

The axiomatic approach used to classify scale spaces was also used in [8] in order to classify interpolation operators according to a set of structural requirements and invariances. Examples are given by the Laplace equation, the AMLE, or the interpolation of level lines by straight lines (related to inpainting/disocclusion [30, 29]). This approach was later extended to image interpolation on surfaces in [7].

Our purpose in this paper is to define a multiscale comparison of images defined on Riemannian manifolds. Given two images u, v defined in their respective image domains (assume \mathbb{R}^2 for simplicity), we want to compare their neighborhoods at the points $x, y \in \mathbb{R}^2$, respectively. The simplest way to compare them would be to compare the two neighborhoods of x, y using the Euclidean distance. That is, let us define

$$(1.1) \quad \mathcal{D}(t, x, y) = \int_{\mathbb{R}^2} g_t(h)(u(x+h) - v(y+h))^2 dh,$$

where g_t is a given window that we assume to be Gaussian of variance t . This formula gives an explicit comparison and assumes that the image domain is the Euclidean plane. Let us note at this point that we could have also used the integral of $u(x+h)v(y+h)$ as a comparison measure. Our purpose is to define such measures in the case of images defined on Riemannian manifolds (e.g., the image plane endowed with an anisotropic metric, such as the structure tensor [46, 47, 4, 3, 35]). It will be shown that these measures are given by the solution of a degenerate elliptic PDE in the variables (x, y) . Unfortunately, in general, it may not always be possible to write this solution as an explicit formula like (1.1). Let us mention at this point that (1.1) is

not an exception; it solves the equation

$$(1.2) \quad \mathcal{D}_t = \Delta_x \mathcal{D} + 2\text{Trace}(D_{xy}^2 \mathcal{D}) + \Delta_y \mathcal{D},$$

which is possibly the simplest case of a linear PDE expressing the multiscale comparison of two image patches. In the case of comparing image patches defined on Riemannian manifolds, there will appear a large family of possibilities, derived from the axiomatic approach. As in [1, 5], the set of axioms will include architectural axioms and a comparison principle that permit us to define multiscale analyses as solutions of a degenerate parabolic PDE. Further specification can be attained by including linear or morphological assumptions. The inclusion of geometric invariances will be subsumed under the requirement of intrinsic definition of the multiscale analysis, independent of the parameterization of the manifold. This essentially restricts the invariances to rotation invariance in the tangent plane. The consideration of other geometric invariance (translation or rotation) will be discussed separately for images defined in \mathbb{R}^N , out of the general classification.

One of the examples of linear multiscale analysis of a similarity measure is the model

$$(1.3) \quad \mathcal{D}_t = \text{Tr}(G_1(x)^{-1} D_x^2 \mathcal{D}) + 2\text{Tr}(G_1(x)^{-1/2} G_2(x)^{-1/2} D_{xy}^2 \mathcal{D}) + \text{Tr}(G_2(y)^{-1} D_y^2 \mathcal{D}),$$

where $\mathcal{M}_i = (\mathbb{R}^N, G_i(x))$, $i = 1, 2$, are two Riemannian manifolds, and $x \in \mathcal{M}_1$, $y \in \mathcal{M}_2$. In particular, if we assume that the metrics are constant in both images, then the model becomes

$$(1.4) \quad D_t = \text{Trace}(A^t A D_x^2 D) + 2\text{Trace}(A B^t D_{xy} D) + \text{Trace}(B^t B D_y^2 D),$$

where A, B are two $N \times N$ matrices. The multiscale similarity measure $C(t, x, y) = \int_{\mathbb{R}^N} g_t(z) C(0, x + Ah, y + Bh) dh$, where g_t is the Gaussian of scale t , and $C(0, x, y) = (I(x) - J(y))^2$, satisfies (4.26).

Let us finally say that from the mathematical point of view the basic ingredients are the papers [1, 8, 7, 5], and our results are an extension of them.

This paper contains mostly the theoretical results that define multiscale analysis for image comparison. From the analysis we will single out several examples, mostly linear examples, and the case of some morphological scale spaces whose complexity is much higher. The use of these comparison measures (distances) for the purpose of computing disparities, correspondences, or determining the most similar patch will be the object of a subsequent paper [10]. We include in the last section of the present paper a preliminary result illustrating the comparison measure of the example above (see also Remark 13).

Let us finally summarize the plan of the paper. In section 2 we collect some basic notation and definitions about Riemannian manifolds. In section 3 we define the basic set of axioms satisfied by multiscale analyses for image similarity measures defined on Riemannian manifolds, and we express them in terms of solutions of an (eventually degenerate) parabolic equation. In section 4 we consider the case of linear multiscale analyses, naturally obtaining that they are expressed as solutions of a linear equation generalizing the case of (1.2). In section 4.1 we will specify our study in the case of \mathbb{R}^N , and the conformal case in \mathbb{R}^N will be studied in section 4.2 (the Euclidean case will be the object of section 4.3). Finally, in section 5 we consider multiscale analyses for image similarity measures that commute with contrast changes, leading to morphological scale spaces that are expressed by functions of curvature operators. In this case, their interpretation is much more complex because it probably reflects the correlations between directions of level lines of both image patches under comparison.

2. Preliminaries. We collect in this section some basic notation and definitions about Riemannian manifolds.

Let (\mathcal{N}, h) be a smooth Riemannian manifold in \mathbb{R}^{N+1} . As a particular case we can consider $\mathcal{N} = \mathbb{R}^N$ (or a domain in \mathbb{R}^N) endowed with a general metric h_{ij} . As usual, given a point $\eta \in \mathcal{N}$, we denote by $T_\eta\mathcal{N}$ the tangent space to \mathcal{N} at the point η . By $T_\eta^*\mathcal{N}$ we denote its dual space.

Let η be a point on \mathcal{N} , let $U \subseteq \mathbb{R}^N$ be an open set containing 0, and let $\psi : U \rightarrow \mathcal{N}$ be any coordinate system such that $\psi(0) = \eta$. Let $h_{ij}(\eta)$ and $\Gamma_{ij}^{\mathcal{N},k}(\eta)$ (indices i, j, k run from 1 to N) denote, respectively, the coefficients of the first fundamental form of \mathcal{N} and the Christoffel symbol computed in the coordinate system ψ around η . For simplicity we shall denote by $H(\eta)$ the (symmetric) matrix $(h_{ij}(\eta))$ and by $\Gamma^{\mathcal{N},k}(\eta)$ the matrix formed by the coefficients $(\Gamma_{ij}^{\mathcal{N},k}(\eta))$, $i, j, k = 1, \dots, N$. We shall use Einstein's convention that repeated indices are summed, and we denote $(a, b) = a_i b^i$.

The scalar product of two vectors $v, w \in T_\eta\mathcal{N}$ will be denoted by $\langle v, w \rangle_\eta$, and the action of a covector $p^* \in T_\eta^*\mathcal{N}$, on a vector $v \in T_\eta\mathcal{N}$, will be denoted by (p^*, v) . Let $\psi : U \rightarrow \mathcal{N}$ be a coordinate system such that $\psi(0) = \eta$, and $h_{ij}(\eta)$ are the coefficients of the first fundamental form at $\eta \in \mathcal{N}$ in ψ . Then, if $v, w \in T_\eta\mathcal{N}$, we have $\langle v, w \rangle_\xi = h_{ij}(\eta)v^i w^j$, where v^i, w^i are the coordinates of v, w in the basis $\frac{\partial}{\partial x^i}|_\eta$ of $T_\eta\mathcal{N}$. Using this basis for $T_\eta\mathcal{N}$ and the dual basis on $T_\eta^*\mathcal{N}$, if $p^* \in T_\eta^*\mathcal{N}$ and $v \in T_\eta\mathcal{N}$, we have $(p^*, v) = p_i v^i$. Notice that we may write $(p^*, v) = h_{ij}(\eta)p^i v^j$, where p^i are the coordinates of the vector p associated to the covector p^* . The relation between both coordinates is given by

$$(2.1) \quad p_i = h_{ij}(\eta)p^j \quad \text{or} \quad p^i = h^{ij}(\eta)p_j,$$

where $h^{ij}(\eta)$ denotes the coefficients of the inverse matrix of $h_{ij}(\eta)$. By a slight abuse of notation, we shall write (2.1) as

$$p^* = Hp \quad \text{or} \quad p = H^{-1}p^*.$$

In this way $H : T_\eta\mathcal{N} \rightarrow T_\eta^*\mathcal{N}$. In the case that ψ is a geodesic coordinate system, the matrix H is the identity matrix $I = (\delta_{ij})$, and I maps vectors to covectors, i.e., $I : T_\eta\mathcal{N} \rightarrow T_\eta^*\mathcal{N}$ (with the same coordinates in the dual basis). We shall denote by I^{-1} the inverse of I , mapping covectors to vectors.

If $U \subseteq \mathbb{R}^N$, and $\psi : U \rightarrow \mathcal{N}$ is a coordinate system with $\psi(0) = \eta$, then $\psi \circ d\psi(0)^{-1} : U' \subseteq T_\eta\mathcal{N} \rightarrow \mathcal{N}$ is a new coordinate system. If we identify T_0U with \mathbb{R}^N and $\{e_i\}$ denotes its canonical basis, then $e'_i = d\psi(0)e_i$ satisfy $\langle e'_i, e'_j \rangle = h_{ij}(\eta)$. From now on, we shall use this identification; thus we shall interpret that any coordinate system around a point $\eta \in \mathcal{N}$ is defined on a neighborhood of 0 in the tangent space $T_\eta\mathcal{N}$.

Maps. Symmetric maps. Quadratic forms. We shall also use this coordinate system to express a bilinear map $\hat{A} : T_\eta\mathcal{N} \times T_\eta\mathcal{N} \rightarrow \mathbb{R}$. Indeed, if (A_{ij}) is the matrix of \hat{A} in this basis, and $v, w \in T_\eta\mathcal{N}$, we may write $\hat{A}(v, w) = A_{ij}v^j w^i$. If $A_j^i = h^{ik}(\eta)A_{kj}$, then A_j^i determines a map called $A : T_\eta\mathcal{N} \rightarrow T_\eta\mathcal{N}$ such that $\hat{A}(v, w) = \langle Av, w \rangle = (HAv, w)$. Observe that $H(\eta)A : T_\eta\mathcal{N} \rightarrow T_\eta^*\mathcal{N}$. Observe also that our notation A_j^i already indicates that $A = (A_j^i)$ maps vectors to vectors. In our notation, we shall not distinguish between matrices and maps.

As usual, we say that a linear map $C : T_\eta\mathcal{N} \rightarrow T_\eta^*\mathcal{N}$ is symmetric if $(Cv, w) = (Cw, v)$ for any $v \in T_\eta\mathcal{N}$, $w \in T_\eta\mathcal{N}$. From now on, we shall use the notation

$$SM_\eta(\mathcal{N}) := \{A : T_\eta\mathcal{N} \rightarrow T_\eta^*\mathcal{N}, A \text{ is symmetric}\}.$$

We shall also write

$$S_\eta(\mathcal{N}) := \{A : T_\eta\mathcal{N} \rightarrow T_\eta\mathcal{N}, H(\eta)A \in SM_\eta(\mathcal{N})\}.$$

If we want to stress that $H(\eta)$ is the metric in $T_\eta\mathcal{N}$, we shall write $(T_\eta\mathcal{N}, H(\eta))$ and denote $SM_\eta(\mathcal{N}, H)$, $S_\eta(\mathcal{N}, H)$ instead of $SM_\eta(\mathcal{N})$, $S_\eta(\mathcal{N})$, respectively.

If $A \in S_\eta(\mathcal{N})$, $v \in T_\eta\mathcal{N}$, $c \in \mathbb{R}$, we define the quadratic polynomial

$$(2.2) \quad Q(x) = \frac{1}{2}\langle Ax, x \rangle + \langle v, x \rangle + c, \quad x \in T_\eta\mathcal{N}.$$

Note that

$$(2.3) \quad Q(x) = \frac{1}{2}(A'x, x) + (p, x) + c, \quad x \in T_\eta\mathcal{N},$$

where $A' = H(\eta)A \in SM_\eta(\mathcal{N})$, $p = H(\eta)v \in T_\eta^*\mathcal{N}$.

Notice that if $A : T_\eta\mathcal{N} \rightarrow T_\eta\mathcal{N}$, we define $A^t : T_\eta^*\mathcal{N} \rightarrow T_\eta^*\mathcal{N}$ by

$$(A^t p, v) = (p, Av) \quad \forall v \in T_\eta\mathcal{N}, p \in T_\eta^*\mathcal{N}.$$

We define $A^{t,h} : T_\eta\mathcal{N} \rightarrow T_\eta\mathcal{N}$ by

$$\langle A^{t,h}v, w \rangle = \langle v, Aw \rangle \quad \forall v, w \in T_\eta\mathcal{N}.$$

From now on, when the point $\eta \in \mathcal{N}$ is understood, we write H instead of $H(\eta)$. Notice that $HA^{t,h} = A^tH$.

If $A \in S_\eta(\mathcal{N})$, then $HA \in SM_\eta(\mathcal{N})$ and $(HA v, w) = (v, HAw)$; that is, $\langle Av, w \rangle = \langle v, Aw \rangle$. That is, $A^{t,h} = A$.

Rotations in the tangent space. Let us define a rotation $R : T_\eta\mathcal{N} \rightarrow T_\eta\mathcal{N}$ as a linear map that satisfies

$$\langle Rv, Rw \rangle = \langle v, w \rangle \quad \forall v, w \in T_\eta\mathcal{N}.$$

Notice that rotations satisfy

$$R^t H R = H.$$

Note also that isometries (rotations) satisfy $R^{t,h} = R^{-1}$.

Let $B : T_\eta\mathcal{N} \rightarrow T_\eta\mathcal{N}$ be a matrix such that $BI^{-1}B^t = H^{-1}$. Thus $B^t H B = I$, and B is mapping an orthonormal basis of $(T_\eta\mathcal{N}, I)$ to an orthonormal basis of $(T_\eta\mathcal{N}, H(\eta))$.

If $R : T_\eta\mathcal{N} \rightarrow T_\eta\mathcal{N}$ is a rotation, then

$$(B^{-1}RB)^{t,I} B^{-1}RB = I.$$

That is, $B^{-1}RB$ is a classical rotation.

Gradient and Hessian. Given a function u on \mathcal{N} , let us denote by $D_{\mathcal{N}}u$ and $D_{\mathcal{N}}^2u$ the gradient and Hessian of u , respectively. In a coordinate system $D_{\mathcal{N}}u$ is the covector $\frac{\partial u}{\partial x^i}$, and $D_{\mathcal{N}}^2u$ is the matrix $\frac{\partial^2 u}{\partial x^i \partial x^j} - \Gamma_{ij}^{\mathcal{N},k} \frac{\partial u}{\partial x^k}$ which acts on tangent vectors. Thus, with this notation $D_{\mathcal{N}}^2u(\eta) : T_\eta\mathcal{N} \times T_\eta\mathcal{N} \rightarrow \mathbb{R}$ is a bilinear map ($\eta \in \mathcal{N}$) and is a symmetric matrix in coordinates. Let us write $\nabla_{\mathcal{N}}u$, the vector of coordinates $h^{ij} \frac{\partial u}{\partial x^j}$. Then $|\nabla_{\mathcal{N}}u(\eta)|_\eta^2 = \langle \nabla_{\mathcal{N}}u(\eta), \nabla_{\mathcal{N}}u(\eta) \rangle_\eta$. To simplify our notation we shall write Du

and ∇u instead of $D_{\mathcal{N}}u$ and $\nabla_{\mathcal{N}}u$. The vector field ∇u satisfies $\langle \nabla u, v \rangle_{\eta} = du(v)$, $v \in T_{\eta}\mathcal{N}$, du being the differential of u .

The manifold $\mathcal{N} = \mathcal{M}^1 \times \mathcal{M}^2$. Let (\mathcal{M}^i, g^i) be a smooth Riemannian manifold with metric g^i , $i = 1, 2$. Let $\Gamma^{(i)}$ be the connection on \mathcal{M}^i . We shall work here with a manifold $\mathcal{N} = \mathcal{M}^1 \times \mathcal{M}^2$ with the metric $h = g^1 \times g^2$, so that $T_{\xi}\mathcal{N} = T_{\xi_1}\mathcal{M}^1 \times T_{\xi_2}\mathcal{M}^2$, $\xi = (\xi_1, \xi_2) \in \mathcal{M}^1 \times \mathcal{M}^2$. If $(v_i, w_i) \in T_{\xi_1}\mathcal{M}^1 \times T_{\xi_2}\mathcal{M}^2$, $\xi = (\xi_1, \xi_2) \in \mathcal{M}^1 \times \mathcal{M}^2$, then we consider the metric

$$\langle (v_1, w_1), (v_2, w_2) \rangle_{\xi} = \langle v_1, v_2 \rangle_{\xi_1} + \langle w_1, w_2 \rangle_{\xi_2} = (G^1(\xi_1)v_1, v_2) + (G^2(\xi_2)w_1, w_2).$$

With a slight abuse of notation, let us write $G(\xi) = \text{diag}(G^1(\xi_1), G^2(\xi_2))$ instead of $H(\xi)$.

Let $\xi = (\xi_1, \xi_2) \in \mathcal{M}^1 \times \mathcal{M}^2$. Let us consider a coordinate system of the form $\psi = (\psi_1, \psi_2) : U_1 \times U_2 \rightarrow \mathcal{M}^1 \times \mathcal{M}^2$ with $\psi_i(0) = \xi_i$, U_i being a neighborhood of 0 in \mathbb{R}^N . Write $x \in U_1, y \in U_2$. Let us denote the connection on $\mathcal{M}^1 \times \mathcal{M}^2$ as $\Gamma := \Gamma^{(1)} \otimes \Gamma^{(2)}$ with indices $i, j, k \in \{1, \dots, 2N\}$ with $\xi_i = \xi_{1i}$, $i \in \{1, \dots, N\}$, and $\xi_i = \xi_{2(i-N)}$, $i \in \{N+1, \dots, 2N\}$. Denote the coordinates as z^i , $i \in \{1, \dots, 2N\}$, with $z^i = x^i$, $i \in \{1, \dots, N\}$, and $z^i = y^{i-N}$, $i \in \{N+1, \dots, 2N\}$. Using the formula

$$(\Gamma^{(1)} \otimes \Gamma^{(2)})_{ij}^k = \frac{1}{2}h^{kl} \left(\frac{\partial h_{jl}}{\partial z^i} + \frac{\partial h_{il}}{\partial z^j} - \frac{\partial h_{ij}}{\partial z^l} \right),$$

we obtain

$$(\Gamma^{(1)} \otimes \Gamma^{(2)})^k(x, y) = \begin{pmatrix} \Gamma^{(1)k}(x) & 0 \\ 0 & \Gamma^{(2)k}(y) \end{pmatrix}.$$

We denote by $SM_{\xi}(\mathcal{N})$ the set of symmetric matrices of size $2N \times 2N$ in $\mathcal{N} = \mathcal{M}^1 \times \mathcal{M}^2$.

A priori connections on $\mathcal{N} = \mathcal{M}^1 \times \mathcal{M}^2$. This is an important concept in this paper and we need to clarify it. Suppose that both manifolds \mathcal{M}^1 and \mathcal{M}^2 coincide with \mathbb{R}^N endowed with the Euclidean metric. Let u, v be two given images in \mathbb{R}^N . Then it would be standard to use the L^2 distance to compare the patches centered at x and y ,

$$(2.4) \quad \mathcal{D}(t, x, y) = \int_{\mathbb{R}^N} g_t(h)(u(x+h) - v(y+h))^2 dh,$$

where g_t is a given window that we assume to be Gaussian of variance t . But if the image v is rotated, we could also use the L^2 distance between u and a rotated version of v (around y), namely,

$$(2.5) \quad \mathcal{D}(t, x, y) = \int_{\mathbb{R}^N} g_t(h)(u(x+h) - v(y+Rh))^2 dh.$$

We admit that this decision is taken a priori and is done thanks to an operator that connects the tangent plane at both points.

Let $\xi = (\xi_1, \xi_2) \in \mathcal{N} = \mathcal{M}_1 \times \mathcal{M}_2$. Let us consider a coordinate system of the form $\psi = (\psi_1, \psi_2) : U_1 \times U_2 \rightarrow \mathcal{M}^1 \times \mathcal{M}^2$ with $\psi_i(0) = \xi_i$, U_i being a neighborhood of 0 in \mathbb{R}^N .

DEFINITION 2.1. *We say that $P(\xi)$, $\xi = (\xi_1, \xi_2) \in \mathcal{N}$, is an a priori connection map in \mathcal{N} if $P(\xi) : (T_{\xi_1}\mathcal{M}^1, G^1(\xi_1)) \rightarrow (T_{\xi_2}\mathcal{M}^2, G^2(\xi_2))$ is an isometry, i.e.,*

$$\langle P(\xi)v, P(\xi)w \rangle_{G^2(\xi_2)} = \langle v, w \rangle_{G^1(\xi_1)} \quad \forall v, w \in T_{\xi_1}\mathcal{M},$$

and we assume also that the map is differentiable in ξ .

Given an a priori connection $P(\xi) : (T_{\xi_1}\mathcal{M}^1, G^1(\xi_1)) \rightarrow (T_{\xi_2}\mathcal{M}^2, G^2(\xi_2))$, we can also define its inverse $P(\xi)^{-1} : (T_{\xi_2}\mathcal{M}^2, G^2(\xi_2)) \rightarrow (T_{\xi_1}\mathcal{M}^1, G^1(\xi_1))$. For simplicity, and understanding that the arguments in P say if we go from \mathcal{M}^1 to \mathcal{M}^2 or inversely, we denote $P(\xi_2, \xi_1) = P(\xi_1, \xi_2)^{-1}$, we have

$$(2.6) \quad P(\xi_2, \xi_1)P(\xi_1, \xi_2) = I.$$

Let us note that if the (orientable) manifold $\mathcal{M}^1 = \mathcal{M}^2 = \mathcal{M}$ admits an a priori connection (into itself), then there is a section of the frame bundle (bundle of orthonormal frames). This is equivalent to saying that there is a section of the bundle of reference systems. This is the notion of parallelizable manifolds. The manifolds $(\mathbb{R}^N, g(x))$ are parallelizable. If \mathcal{M} has dimension 2, then \mathcal{M} is parallelizable if and only if its Euler–Poincaré characteristic is 0 [44]. Any orientable manifold of dimension 3 is parallelizable [42].

Remark 1. Note that if we have a complete manifold with empty cut locus, we can define the a priori connection in it by parallel transport without ambiguities.

Remark 2. Note that if $P(\xi)$ is an a priori connection and we give two maps $R : \mathcal{M}^1 \rightarrow \text{Isom}(T\mathcal{M}^1)$, $\bar{R} : \mathcal{M}^2 \rightarrow \text{Isom}(T\mathcal{M}^2)$, then $\bar{R}(\xi_2)P(\xi)R(\xi_1)$ is also an a priori connection.

In coordinates, $P(\xi)$ expresses the a priori connection in the coordinate system $\psi_1 \rightarrow \psi_2$. The isometry property can be written as

$$(P(\xi)^t G^2(\xi_2) P(\xi) v, w) = (G^1(\xi_1) v, w),$$

where $P(\xi)$ is expressed in the basis of $T_{\xi_1}\mathcal{M}^1$ associated to the metric $G^1(\xi_1)$ and the basis of $T_{\xi_2}\mathcal{M}^2$ associated to the metric $G^2(\xi_2)$. Then

$$(2.7) \quad P(\xi)^t G^2(\xi_2) P(\xi) = G^1(\xi_1).$$

Let us compute the a priori connection in another coordinate system. Let $\bar{\psi} = (\bar{\psi}_1, \bar{\psi}_2)$ be another coordinate system around ξ . Let $G^i(\xi_i), \bar{G}^i(\xi_i)$, $i = 1, 2$, be the metric matrices represented in the coordinate system $\psi_i, \bar{\psi}_i$, respectively. Let $B_{G^i, \bar{G}^i}(\xi_i) = D(\psi_i^{-1} \circ \bar{\psi}_i)(0)$, $i = 1, 2$, and $B_{G, \bar{G}}(\xi) = (B_{G^1, \bar{G}^1}(\xi_1), B_{G^2, \bar{G}^2}(\xi_2))$. Note that $B_{G^i, \bar{G}^i}(\xi_i) : (T_{\xi_i}\mathcal{M}^i, \bar{G}^i(\xi_i)) \rightarrow (T_{\xi_i}\mathcal{M}^i, G^i(\xi_i))$ is such that

$$(2.8) \quad B_{G^i, \bar{G}^i}(\xi_i)^t G^i(\xi_i) B_{G^i, \bar{G}^i}(\xi_i) = \bar{G}^i(\xi_i).$$

Note also that all matrices here are uniquely defined.

Using (2.8), we express (2.7) as

$$(2.9) \quad P^t(\xi) B_{G^2, \bar{G}^2}(\xi_2)^{-t} \bar{G}^2(\xi_2) B_{G^2, \bar{G}^2}(\xi_2)^{-1} P(\xi) = B_{G^1, \bar{G}^1}(\xi_1)^{-t} \bar{G}^1(\xi_1) B_{G^1, \bar{G}^1}(\xi_1)^{-1}.$$

If we define

$$(2.10) \quad \bar{P}(\xi) := B_{G^2, \bar{G}^2}(\xi_2)^{-1} P(\xi) B_{G^1, \bar{G}^1}(\xi_1),$$

then $\bar{P}(\xi)$ is an a priori connection in the coordinate system $\bar{\psi}_1 \rightarrow \bar{\psi}_2$, $\bar{P}(\xi) : (T_{\xi_1}\mathcal{M}^1, \bar{G}^1(\xi_1)) \rightarrow (T_{\xi_2}\mathcal{M}^2, \bar{G}^2(\xi_2))$. Indeed, we can express (2.9) as

$$(2.11) \quad \bar{P}^t(\xi) \bar{G}^2(\xi_2) \bar{P}(\xi) = \bar{G}^1(\xi_1),$$

which is the isometry property defining a priori connections. We say that $\overline{P}(\xi)$ is the derived a priori connection from $P(\xi)$ and $\overline{\psi}$.

We can rewrite (2.10) as

$$(2.12) \quad B_{G^2, \overline{G}^2}(\xi_2) \overline{P}(\xi) = P(\xi) B_{G^1, \overline{G}^1}(\xi_1),$$

and we see that both maps $B_{G^1, \overline{G}^1}(\xi_1)$ and $B_{G^2, \overline{G}^2}(\xi_2)$ reflect the same rotation when expressed in the corresponding a priori connections $P(\xi)$ and $\overline{P}(\xi)$, respectively.

DEFINITION 2.2. *We say that the coordinate systems $\psi, \overline{\psi}$ are $P(\xi)$ -related if $\overline{P}(\xi)$ is defined by (2.10). We will also say that they are R -related.*

Let us consider the case where $\mathcal{M}^1 = \mathcal{M}^2 = \mathcal{M}$ and $P(\xi)$ is an internal a priori connection given from parallel transport between ξ_1 and ξ_2 , which is an isometry. Then one can define $\overline{P}(\xi)$ by parallel transport expressed in the coordinate systems $\overline{\psi}_1, \overline{\psi}_2$.

Generation of a priori connections. Fix a geodesic coordinate system around each point of \mathcal{M}^i . $I^i(\xi_i)$ is referred to this system for each $\xi_i \in \mathcal{M}^i$. For each $\xi \in \mathcal{N}$, let us consider an isometry map (assuming that it exists)

$$Q(\xi) : (T_{\xi_1} \mathcal{M}^1, I^1(\xi_1)) \rightarrow (T_{\xi_2} \mathcal{M}^2, I^2(\xi_2)).$$

Call $\text{Isom}((T\mathcal{M}^1, I^1), (T\mathcal{M}^2, I^2))$ this set of maps. Let us note that this is nothing else than an a priori connection. We just have one concept, and we express it in different coordinate systems.

Thus what we are going to do is to give the a priori connection Q in a geodesic coordinate field \mathcal{GS} and derive its expression in another coordinate system field.

Let $B^i(\xi_i) : (T_{\xi_i} \mathcal{M}^i, I^i(\xi_i)) \rightarrow (T_{\xi_i} \mathcal{M}^i, G^i(\xi_i))$ be the corresponding canonical maps connecting a geodesic coordinate system \mathcal{GS} around ξ_i to $(T_{\xi_i} \mathcal{M}^i, G^i(\xi_i))$. Thus

$$B^i(\xi_i)^t G^i(\xi_i) B^i(\xi_i) = I^i(\xi_i).$$

Note that the map $B^i(\xi_i)$ is uniquely defined by the coordinate systems. Changing (rotating) the geodesic coordinate system, we get a different matrix.

Let $Q(\xi) \in \text{Isom}((T\mathcal{M}^1, I^1), (T\mathcal{M}^2, I^2))$, where each I is referred to \mathcal{GS} , and let us define

$$P(\xi) := B^2(\xi_2) Q(\xi) B^1(\xi_1)^{-1}.$$

Then $P(\xi)$ is an a priori connection map.

Let $\overline{\mathcal{GS}}$ be another geodesic coordinate system field. Let $\overline{Q}(\xi) : \text{Isom}((T\mathcal{M}^1, I^1) \rightarrow (T\mathcal{M}^2, I^2))$, where each I is referred to $\overline{\mathcal{GS}}$. Let $\overline{B}^i(\xi_i) : (T_{\xi_i} \mathcal{M}^i, I^i(\xi_i)) \rightarrow (T_{\xi_i} \mathcal{M}^i, \overline{G}^i(\xi_i))$ be the corresponding map connecting a geodesic coordinate system $\overline{\mathcal{GS}}$ around ξ_i to $(T_{\xi_i} \mathcal{M}^i, \overline{G}^i(\xi_i))$. Thus

$$\overline{B}^i(\xi_i)^t \overline{G}^i(\xi_i) \overline{B}^i(\xi_i) = I^i(\xi_i).$$

Note that the map $\overline{B}^i(\xi_i)$ is uniquely defined by the coordinate systems. Changing (rotating) the geodesic coordinate system, we get a different matrix. Note that from the above identities, it is easy to check that $B^i(\xi_i)^{-1} B_{G^i, \overline{G}^i}(\xi_i) \overline{B}^i(\xi_i) : (T_{\xi_i} \mathcal{M}^i, I^i(\xi_i)) \rightarrow (T_{\xi_i} \mathcal{M}^i, I^i(\xi_i))$ is a (classical) rotation.

Let us define

$$\overline{P}(\xi) := \overline{B}^2(\xi_2) \overline{Q}(\xi) \overline{B}^1(\xi_1)^{-1}.$$

Then $\overline{P}(\xi)$ is an a priori connection map.

Let us express that $\overline{P}(\xi)$ is the derived connection from $P(\xi)$ in terms of $Q(\xi)$ and $\overline{Q}(\xi)$. Indeed, $\overline{P}(\xi)$ is the derived connection from $P(\xi)$ and is written as (2.10). Introducing the definitions of $P(\xi)$ and $\overline{P}(\xi)$ into (2.10), we see that $\overline{Q}(\xi)$ is derived from $Q(\xi)$ if and only if

$$(2.13) \quad \overline{Q}(\xi) = [B^2(\xi_2)^{-1}B_{G^2, \overline{G}^2}(\xi_2)\overline{B}^2(\xi_2)]^{-1}Q(\xi)[B^1(\xi_1)^{-1}B_{G^1, \overline{G}^1}(\xi_1)\overline{B}^1(\xi_1)].$$

One interprets this by saying that $[B^2(\xi_2)^{-1}B_{G^2, \overline{G}^2}(\xi_2)\overline{B}^2(\xi_2)]$ expresses the same Euclidean rotation in different coordinate systems.

When is $Q(\xi) = \overline{Q}(\xi)$? It is when we use the same coordinate systems for the identity maps appearing in $B^i(\xi_i) : (T_{\xi_i}\mathcal{M}^i, I^i(\xi_i)) \rightarrow (T_{\xi_i}\mathcal{M}^i, G^i(\xi_i))$ and in $\overline{B}^i(\xi_i) : (T_{\xi_i}\mathcal{M}^i, I^i(\xi_i)) \rightarrow (T_{\xi_i}\mathcal{M}^i, \overline{G}^i(\xi_i))$, that is, when $\mathcal{G}\mathcal{S} = \overline{\mathcal{G}\mathcal{S}}$.

Related rotations. Let us consider a coordinate system field and an a priori connection $P(\xi) : (T_{\xi_1}\mathcal{M}^1, G^1(\xi_1)) \rightarrow (T_{\xi_2}\mathcal{M}^2, G^2(\xi_2))$ in that system field. Let us consider a second coordinate system field with metric $\overline{G}^i(\xi_i) = G^i(\xi_i)$, $i = 1, 2$, for each $\xi_i \in \mathcal{M}^i$ so that $B_{G^i, \overline{G}^i}(\xi_i)$ is an isometry field. Let $R^i(\xi_i) := B_{G^i, \overline{G}^i}(\xi_i)$. Let $\overline{P}(\xi)$ be the derived connection. Then (2.12) can be written as

$$(2.14) \quad R^2(\xi_2) = P(\xi)R^1(\xi_1)\overline{P}(\xi)^{-1}.$$

We say that $(R^1(\xi_1), R^2(\xi_2))$ are P -related or R -related, and we call $R = (R^1(\xi_1), R^2(\xi_2))$ a diagonally related rotation (or just a diagonal rotation if no confusion arises).

Related germs of functions on $\mathcal{N} = \mathcal{M}^1 \times \mathcal{M}^2$. Let $C_b(\mathcal{N})$ denote the space of bounded continuous functions in \mathcal{N} with the maximum norm. We think of $C_b(\mathcal{N})$ as the space of similarity functions on $\mathcal{N} = \mathcal{M}^1 \times \mathcal{M}^2$. We denote by $C_b^\infty(\mathcal{N})$ the space of infinitely differentiable functions on \mathcal{N} .

Let $C \in C_b(\mathcal{N})$. Let us denote

$$(C, \psi)(x, y) = C(\psi_1(x), \psi_2(y)) \quad \forall (x, y) \in U_1 \times U_2.$$

Thus, we can say that $\psi = (\psi_1, \psi_2)$ and $\overline{\psi} = (\overline{\psi}_1, \overline{\psi}_2)$ are R -related if (2.14) holds. If $\overline{\psi}$ is R -related to ψ , we write $(C, \overline{\psi})$ as $R(C, \psi)$. Note that $R(C, \psi)$ is a linear map for the restriction of functions in $C_b(\mathcal{N})$ to a neighborhood of $(0, 0)$.

Gradient and Hessian. We denote by $SM_\xi(\mathcal{N})$ the set of symmetric matrices of size $2N \times 2N$ in $\mathcal{N} = \mathcal{M}^1 \times \mathcal{M}^2$. In coordinates, we denote $D_{\mathcal{N}}C = (D_x C, D_y C)$ by

$$D_{\mathcal{N}}^2 C = \begin{pmatrix} D_{\mathcal{N}, xx} C & D_{\mathcal{N}, xy} C \\ D_{\mathcal{N}, xy} C & D_{\mathcal{N}, yy} C \end{pmatrix}.$$

In coordinates, with $i, j, k \in \{1, \dots, N\}$,

$$D_{\mathcal{N}}^2 C = \begin{pmatrix} \frac{\partial^2 C}{\partial x^i \partial x^j} & \frac{\partial^2 C}{\partial x^i \partial y^j} \\ \frac{\partial^2 C}{\partial y^j \partial x^i} & \frac{\partial^2 C}{\partial y^j \partial y^i} \end{pmatrix} - \begin{pmatrix} \Gamma^{(1)k}(x) \frac{\partial C}{\partial x^k} & 0 \\ 0 & \Gamma^{(2)k}(y) \frac{\partial C}{\partial y^k} \end{pmatrix}.$$

3. Multiscale analysis of image similarity measures. For simplicity, we shall write $\mathcal{N} = \mathcal{M}^1 \times \mathcal{M}^2$. The metric will be denoted by $g = g^1 \times g^2$, and $G(\xi), G^1(\xi_1), G^2(\xi_2)$ will be the corresponding matrices, $\xi = (\xi_1, \xi_2) \in \mathcal{N}$. Let $(\kappa) := \kappa_n$ be an increasing sequence of nonnegative constants.

$$\mathcal{Q}((\kappa)) := \{C \in C_b^\infty(\mathcal{N}) : \|D^\alpha C\|_\infty \leq \kappa_n \quad \forall n \geq 0 \quad \forall |\alpha| \leq n\}.$$

As usual, $O(f)$ (resp., $o(f)$) will denote any expression which is bounded by $C|f|$ for some constant $C > 0$ (resp., such that $\frac{o(f)}{|f|} \rightarrow 0$ as $f \rightarrow 0$). Assume that $T_t : C_b(\mathcal{N}) \rightarrow C_b(\mathcal{N})$ is a nonlinear operator for any $t \geq 0$. We shall denote $C(t, x, y) = T_t C(x, y)$, $C \in C_b(\mathcal{N})$. Assume that we are given an a priori connection P on \mathcal{N} .

Our motivation for the proposed set of axioms is the same as in the pioneering work of [1], to which we refer the reader for a detailed justification.

Architectural axioms.

$$[\text{Recursivity}] \quad T_0(C) = C, T_s(T_t C) = T_{s+t} C \quad \forall s, t \geq 0, \forall C \in C_b(\mathcal{N}).$$

The recursivity axiom is a strong version of causality which implies that the similarity measure at a coarser scale can be deduced from a finer one, which is a natural property in image analysis and a sound hypothesis in human vision [1].

$$[\text{Infinitesimal generator}] \quad \frac{T_h(C, \psi)(\xi) - (C, \psi)(\xi)}{h} \rightarrow (\mathcal{A}(C), \psi) \quad \text{as } h \rightarrow 0+$$

for any $C \in C_b^\infty(\mathcal{N})$ and any coordinate system $\psi = (\psi_1, \psi_2)$ around ξ . We assume that

$$(3.1) \quad T_t(R(C, \psi))(\xi) = R(T_t(C), \psi)(\xi) + o(t) = T_t(C)(\xi) + o(t) \quad \text{as } t \rightarrow 0+$$

for any $C \in C_b(\mathcal{N})$, any coordinate system $\psi = (\psi_1, \psi_2)$, and any R 's which are P -related rotations. We have denoted by $R(C, \psi)$ the function in the coordinate system ψ which is $P(\xi)$ -related (or R -related) to ψ .

Writing (3.1) in terms of the generator \mathcal{A} , we have

$$\begin{aligned} R(C, \psi)(0) + t\mathcal{A}(R(C, \psi))(0) + o(t) &= R((C, \psi) + t\mathcal{A}(C, \psi))(0) + o(t) \\ &= C(\xi) + t\mathcal{A}(C, \psi)(0) + o(t). \end{aligned}$$

Using the linearity of $R(C, \psi)$, dividing by t , and letting $t \rightarrow 0+$, we obtain

$$(3.2) \quad \mathcal{A}(R(C, \psi))(0) = R\mathcal{A}(C, \psi)(0) = \mathcal{A}(C, \psi)(0)$$

for any $C \in C_b(\mathcal{N})$, any coordinate system $\psi = (\psi_1, \psi_2)$, and any R P -related rotations.

Remark 3. In $T_t(R(C, \psi))(\xi)$ the a priori connection is expressed in the coordinate system $\bar{\psi} = (\bar{\psi}_1, \bar{\psi}_2)$. In $R(T_t(C), \psi)(\xi) = (T_t(C), \psi)(\xi)$ the a priori connection is expressed in the coordinate system $\psi = (\psi_1, \psi_2)$. That is, the infinitesimal generator axiom says that both expressions are the same (intrinsic character of T_t) when the coordinate systems are R -related.

Remark 4. The infinitesimal generator axiom contains the invariance with respect to diagonal rotations in the tangent plane of $\mathcal{M}^1 \times \mathcal{M}^2$. When $(\mathcal{M}^i, G^i) = (\mathcal{M}, G) = (\mathbb{R}^N, I)$ it amounts to invariance with respect to Euclidean diagonal rotations in \mathbb{R}^{2N} . That is, $T_t(RC) = RT_t(C)$ for all $t \geq 0$, for all $C \in C_b^\infty(\mathbb{R}^N \times \mathbb{R}^N)$, and for all $R \in O(N)$ (Euclidean rotations in \mathbb{R}^N), where $RC(x, y) = C(Rx, Ry)$.

Remark 5. When $\mathcal{M} = \mathbb{R}^N$ with the Euclidean metric the axiom is just

$$[\text{Infinitesimal generator}] \quad \frac{T_h C - C}{h} \rightarrow \mathcal{A}(C) \text{ as } h \rightarrow 0+.$$

This holds for any $C \in C_b^\infty(\mathbb{R}^N \times \mathbb{R}^N)$.

In some sense the coordinate system around each point is always the same—the canonical system; they are related by the identity.

$$[\text{Regularity axiom}] \|T_t(C+h\tilde{C})-(T_t(C)+h\tilde{C})\|_\infty \leq Mht \quad \forall h, t \in [0, 1], \forall C, \tilde{C} \in \mathcal{Q}((\kappa)),$$

where the constant M depends on $\mathcal{Q}((\kappa))$.

$$[\text{Locality}] \quad T_t(C)(x) - T_t(\tilde{C})(x) = o(t) \quad \text{as } t \rightarrow 0+, \quad x \in \mathbb{R}^N \quad \forall C, \tilde{C} \in C_b(\mathcal{N})$$

such that $D^\alpha C(x) = D^\alpha \tilde{C}(x)$ for all multi-indices α .

Comparison principle.

$$[\text{Comparison principle}] \quad T_t C \leq T_t \tilde{C} \quad \forall t \geq 0, \quad \forall C, \tilde{C} \in C_b^\infty(\mathcal{N})$$

such that $C \leq \tilde{C}$.

The comparison principle is an order-preserving property. It means that if a similarity measure is always smaller than another, then applying a multiscale analysis does not invert this relation. Intuitively, the multiscale analysis produces low resolution versions of the similarity measures, which should be consistent with the initial ones.

Morphological axioms.

$$[\text{Gray level shift invariance}] \quad T_t(0) = 0, \quad T_t(C + \kappa) = T_t(C) + \kappa \\ \forall t \geq 0, \quad \forall C \in C_b^\infty(\mathcal{N}), \quad \forall \kappa \in \mathbb{R}.$$

$$[\text{Gray scale invariance}] \quad T_t(f(C)) = f(T_t(C)) \quad \forall t \geq 0, \quad \forall C \in C_b^\infty(\mathcal{N}),$$

and for any strictly increasing function $f : \mathbb{R} \rightarrow \mathbb{R}$.

Let us clarify that in this work the morphological axioms are required not for the images but for the similarity measures. We have followed the same terminology as in [1, 5].

THEOREM 3.1. *Let T_t be a multiscale analysis satisfying the recursivity, infinitesimal generator, and regularity axioms. Then $\mathcal{A}(C_r) \rightarrow \mathcal{A}(C)$ in $C_b(\mathcal{N})$ if $C_r, C \in C_b^\infty(\mathcal{N})$ and $D^\alpha C_r \rightarrow D^\alpha C$ in $C_b(\mathcal{N})$ for all α with $|\alpha| \geq 0$.*

The proof follows the same lines of the corresponding result in [1], particularly section 3.1, Theorem 1 (see also Theorem 3.1 in [5]), and so we shall omit it.

The following results were proved in [1] (section 3.2, Theorem 2) for multiscale analysis on images and extended to images on manifolds in [5] (particularly in Theorem 3.2). We follow the presentation in [5].

THEOREM 3.2. *Let T_t be a multiscale analysis satisfying all architectural axioms and the comparison principle. Then there exists a function $F : SM_\xi(\mathcal{N}) \times T_\xi^* \mathcal{N} \times \mathbb{R} \times \mathcal{N} \rightarrow \mathbb{R}$ increasing with respect to its first argument such that*

$$\frac{T_t(C, \psi) - (C, \psi)}{t} \rightarrow F(D^2(C \circ \psi)(0), D(C \circ \psi)(0), C(\xi), \xi, G, \Gamma^k) \text{ in } C_b(\mathcal{N}) \text{ as } t \rightarrow 0+$$

for all $C \in C_b^\infty(\mathcal{N})$, ψ being a coordinate system around $\xi \in \mathcal{N}$. The function F is continuous in its first three arguments. If we assume that T_t is gray level shift invariant, then the function F does not depend on u .

Recall that we have denoted $G = (G^1, G^2)$ and $\Gamma = \Gamma^{(1)} \otimes \Gamma^{(2)}$. Notice that we did not denote explicitly the arguments for G, Γ^k . Notice that the first argument in F is a symmetric map from $T_\xi \mathcal{N}$ to $T_\xi^* \mathcal{N}$.

Remark 6. We could also have written F as a function $\hat{F} : S_\xi(\mathcal{N}) \times T_\xi \mathcal{N} \times \mathbb{R} \times \mathcal{N} \rightarrow \mathbb{R}$, so that $\hat{F}(A, v, c, \xi, G, \Gamma^k) = F(GA, Gv, c, \xi, G, \Gamma^k)$.

Let us make precise our statement that the function F is increasing in its first argument.

LEMMA 3.3. *Let $\xi \in \mathcal{N}$, and let $\psi : U \rightarrow \mathcal{N}$ be a coordinate system around ξ . Let G, Γ^k be the metric coefficients and the Christoffel symbols of \mathcal{N} in the coordinate system ψ at the point ξ . Let $A_1, A_2 : T_\xi \mathcal{N} \rightarrow T_\xi^* \mathcal{N}$ be two matrices such that A_1, A_2 are symmetric, $p \in T_\xi^* \mathcal{N}$, $c \in \mathbb{R}$. If $A_1 \leq A_2$, then*

$$F(A_1, p, c, \xi, G, \Gamma^k) \leq F(A_2, p, c, \xi, G, \Gamma^k).$$

Thus F is elliptic.

THEOREM 3.4. *Let T_t be a multiscale analysis satisfying the all architectural axioms, the comparison principle, and gray level shift invariance. If $C(t, \xi) = T_t C(\xi)$, then u is a viscosity solution of*

$$(3.3) \quad C_t = F(D_N^2 C, DC, \xi, G, \Gamma^k),$$

with $C(0, \xi) = C(\xi)$.

The proof that $C(t, \xi) = T_t C(\xi)$ is the viscosity solution of (3.3) follows as in [1, section 3.2, Theorem 2], [16, Chapters 19 and 20].

The next lemma is crucial in what follows. It relates the matrices and vectors defining a quadratic polynomial in two coordinate systems around a point $\xi \in \mathcal{M}$. For a proof, we refer the reader to [7] (particularly Lemma 2).

LEMMA 3.5. *Let U, \bar{U} be two neighborhoods of 0 in \mathbb{R}^{2N} , and let $\psi : U \rightarrow \mathcal{N}$, $\bar{\psi} : \bar{U} \rightarrow \mathcal{N}$ be two coordinate systems around the point $\xi \in \mathcal{N}$, i.e., $\psi(0) = \xi$, $\bar{\psi}(0) = \xi$. Assume that the change of coordinates $\Psi = \psi^{-1} \circ \bar{\psi} : \bar{U} \rightarrow U$ is a diffeomorphism. Let $G, \Gamma = \Gamma^{(1)} \otimes \Gamma^{(2)}$ (resp., $\bar{G}, \bar{\Gamma} = \bar{\Gamma}^{(1)} \otimes \bar{\Gamma}^{(2)}$) be the metric coefficients and the Christoffel symbols of \mathcal{N} in the coordinate system ψ (resp., $\bar{\psi}$) at the point ξ . Let $Q : U \rightarrow \mathbb{R}$ be the quadratic polynomial*

$$(3.4) \quad Q(v) = \frac{1}{2}(GAv, v) + (p, v) + c.$$

Let $\bar{Q}(\bar{v}) := (Q \circ \Psi)(\bar{v})$. Then $\bar{Q}(\bar{v}) = Q'(\bar{v}) + O(|\bar{v}|^3)$ in a neighborhood of 0, where Q' is the quadratic polynomial

$$(3.5) \quad Q'(\bar{v}) = \frac{1}{2}(\bar{G}B^{-1}AB\bar{v}, \bar{v}) + \frac{1}{2}(\bar{\Gamma}(B^t p)(\bar{v}), \bar{v}) - \frac{1}{2}(B^t \Gamma(p)(B\bar{v}), \bar{v}) + (B^t p, \bar{v}) + c,$$

and $B = D\Psi(0)$.

Note that we have denoted $v \in T_\xi \mathcal{N}$, $p \in (T_\xi \mathcal{N})^*$, $A \in S_\xi(\mathcal{N})$.

We are interested in the application of this lemma when $\psi = (\psi_1, \psi_2)$, $\bar{\psi} = (\bar{\psi}_1, \bar{\psi}_2)$, so that $B = (B^1, B^2)$, where $B^1 = D(\psi_1^{-1} \circ \bar{\psi}_1)(0)$ and $B^2 = D(\psi_2^{-1} \circ \bar{\psi}_2)(0)$. The map B satisfies

$$(3.6) \quad B^t G = \bar{G} B^{-1}.$$

In coordinates, for $i = 1, 2$, we have

$$(3.7) \quad (B^i)^t G^i(\xi_i) = \bar{G}^i(\xi_i)(B^i)^{-1}.$$

PROPOSITION 3.6. *Let T_t be a multiscale analysis on \mathcal{N} satisfying the architectural axioms and the comparison principle. Let $\psi = (\psi_1, \psi_2) : U = U_1 \times U_2 \rightarrow \mathcal{N}$ be a coordinate system around $\xi \in \mathcal{N}$. Let G, Γ be the metric coefficients and the Christoffel symbols of \mathcal{N} in the coordinate system ψ at the point ξ . For any symmetric matrix $X = (X_{ij}) : (T_\xi \mathcal{N}, I) \rightarrow (T_\xi^* \mathcal{N}, I)$ in $SM_\xi(\mathcal{N}, I)$, $q \in (T_\xi^* \mathcal{N}, I)$, and $a \in \mathbb{R}$, let us define the function*

$$(3.8) \quad \mathcal{H}(X, q, a, \xi) = F(X, q, a, \xi, I, 0);$$

that is, \mathcal{H} is the function F obtained when using a geodesic coordinate system. Then

$$(3.9) \quad F(A, p, a, \xi, G, \Gamma^k) = \mathcal{H}(B^t(A - \Gamma(p))B, B^t p, c, \xi)$$

for any matrix $A \in SM_\xi(\mathcal{N})$, and any covector p , where $BB^t = G^{-1}$. Moreover, the function H satisfies

$$(3.10) \quad \mathcal{H}(A', p', c, \xi) = \mathcal{H}(R^t A' R, R^t p', c, \xi),$$

where $A' : (T_\xi \mathcal{N}, I) \rightarrow (T_\xi^* \mathcal{N}, I)$ is any matrix in $SM_\xi(\mathcal{N}, I)$, $p' \in (T_\xi^* \mathcal{N}, I)$, and R is any Euclidean rotation in $(T_\xi \mathcal{N}, I)$ of the form $R = \text{diag}(R_0, R_0)$ where R_0 is an Euclidean rotation in $(T_\xi \mathcal{M}, I)$.

Our notation $BB^t = G^{-1}$ contains a slight abuse of notation, since $B : T_\xi \mathcal{N} \rightarrow T_\xi \mathcal{N}$ and $B^t : T_\xi^* \mathcal{N} \rightarrow T_\xi^* \mathcal{N}$. The correct notation should be $BI^{-1}B^t$.

Although the proof is essentially contained in Proposition 3.6 in [5], the statement is slightly different because rotations in $T_\xi \mathcal{N}$ have a diagonal form. We give the detailed proof in order to clarify this. This is due to our assumption in the infinitesimal generator axiom of the covariance with respect to R -related coordinate systems.

Proof. We use the notation of Lemma 3.5, so that $B = (B_{G^1, \overline{G}^1}(\xi_1), B_{G^2, \overline{G}^2}(\xi_2))$. Note that the quadratic forms Q and \overline{Q} are $P(\xi)$ -related (or R -related). For convenience then we use the symmetric map GA . Since $Q \circ \psi^{-1} = \overline{Q} \circ \overline{\psi}^{-1}$ in $\psi(U) \cap \overline{\psi}(\overline{U})$, with a slight abuse of notation (act T_t on polynomials), using the infinitesimal generator axiom, we have

$$(3.11) \quad \lim_{t \rightarrow 0} \frac{T_t(Q \circ \psi^{-1})(\xi) - Q \circ \psi^{-1}(\xi)}{t} = F(GA, p, c, \xi, G, \Gamma^k)$$

and

$$(3.12) \quad \begin{aligned} & \lim_{t \rightarrow 0} \frac{T_t(\overline{Q} \circ \overline{\psi}^{-1})(\xi) - \overline{Q} \circ \overline{\psi}^{-1}(\xi)}{t} \\ &= F(\overline{G}B^{-1}AB + \overline{\Gamma}(B^t p) - B^t \Gamma(p)B, B^t p, c, \xi, \overline{G}, \overline{\Gamma}^k); \end{aligned}$$

that is, they are the expressions in the corresponding coordinate system, and they coincide since both are $P(\xi)$ -related; i.e., we have

$$(3.13) \quad F(GA, p, c, \xi, G, \Gamma^k) = F(\overline{G}B^{-1}AB + \overline{\Gamma}(B^t p) - B^t \Gamma(p)B, B^t p, c, \xi, \overline{G}, \overline{\Gamma}^k)$$

or, using (3.6),

$$(3.14) \quad F(GA, p, c, \xi, G, \Gamma^k) = F(B^t(GA - \Gamma(p))B + \overline{\Gamma}(B^t p), B^t p, c, \xi, \overline{G}, \overline{\Gamma}^k).$$

Now, for any symmetric matrix $X = (X_{ij}) \in SM_\xi(\mathcal{N})$, any $q \in T_\xi^*\mathcal{N}$, and $a \in \mathbb{R}$, let us define the function \tilde{F} by the identity

$$(3.15) \quad \tilde{F}(X, q, a, \xi, G, \Gamma^k) = F(X + \Gamma(q), q, a, \xi, G, \Gamma^k).$$

In terms of \tilde{F} , (3.14) can be written as

$$(3.16) \quad \tilde{F}(GA - \Gamma(p), p, c, \xi, G, \Gamma^k) = \tilde{F}(B^t(GA - \Gamma(p))B, B^t p, c, \xi, \overline{G}, \overline{\Gamma}^k).$$

By varying the quadratic polynomials, the above equation holds for any matrix $A = (A_j^i)$ such that $GA \in SM_\xi(\mathcal{N})$, any diagonal invertible matrix $B : T_\xi\mathcal{N} \rightarrow T_\xi\mathcal{N}$ (better $B = (B^1(\xi_1), B^2(\xi_2)) = (B_{G^1, \overline{G}^1}(\xi_1), B_{G^2, \overline{G}^2}(\xi_2))$ and $B^i(\xi_i) : (T_\xi\mathcal{M}^i, \overline{G}^i(\xi_i)) \rightarrow (T_\xi\mathcal{M}^i, G^i(\xi_i))$), and any $p \in T_\xi\mathcal{N}$. Here $B^t G = \overline{G} B^{-1}$. This holds in particular for any diagonal rotation $R = (R_1, R_2)$ (related by (2.12), and we use this convention in what follows) in $T_\xi\mathcal{N}$ (so that $R^t G R = G$):

$$(3.17) \quad \tilde{F}(GA - \Gamma(p), p, c, \xi, G, \Gamma^k) = \tilde{F}(R^t(GA - \Gamma(p))R, R^t p, c, \xi, G, \overline{\Gamma}^k).$$

Now, we choose ψ as a geodesic coordinate system around ξ for which $G = I$, and $\Gamma^k = 0$. In this case, (3.6) can be written as $\overline{G} = B^t I B = B^t B$. We may write (3.16) as

$$(3.18) \quad \tilde{F}(IA, p, c, \xi, I, 0) = \tilde{F}(B^t I A B, B^t p, c, \xi, B^t B, \overline{\Gamma}^k),$$

and this identity holds for any symmetric matrix $IA \in SM_\xi(\mathcal{N}, I)$, any vector $p \in T_\xi^*\mathcal{M}^2$, and any diagonal invertible matrix B (the metric $\overline{G} = B^t B$). Once again, we change variables and write $A' = B^t I A B$, $p' = B^t p$, $B' = B^{-1}$. Then we write (3.18) as

$$(3.19) \quad \tilde{F}(A', p', c, \xi, \overline{G}, \overline{\Gamma}^k) = \tilde{F}(B'^t A' B', B'^t p', c, \xi, I, 0),$$

and this identity holds for any symmetric matrix $A' : T_\xi\mathcal{N} \rightarrow T_\xi^*\mathcal{N}$ in $SM_\xi(\mathcal{N}, \overline{G})$, any $p' \in T_\xi^*\mathcal{N}$, and any diagonal invertible matrix $B' : T_\xi\mathcal{N} \rightarrow T_\xi\mathcal{N}$, where $\overline{G} = (B'^t)^{-1} B'^{-1}$. This clearly shows that \tilde{F} does not depend on G and Γ^k in the last two arguments. All its dependence is contained in the first argument. Let us introduce the function \mathcal{H} to make this explicit.

Now, for any symmetric matrix $X = (X_{ij}) : (T_\xi\mathcal{N}, I) \rightarrow (T_\xi^*\mathcal{N}, I)$ in $SM_\xi(\mathcal{N}, I)$, any $q \in (T_\xi^*\mathcal{N}, I)$, and scalar a , let us define the function \mathcal{H} by the identity

$$(3.20) \quad \mathcal{H}(X, q, a, \xi) = \tilde{F}(X, q, a, \xi, I, 0).$$

Note that by (3.15), and (3.20), we have

$$\mathcal{H}(X, q, a, \xi) = F(X, q, a, \xi, I, 0);$$

that is, \mathcal{H} is the function F obtained when using a geodesic coordinate system. Hence, (3.19) can be written as

$$(3.21) \quad \tilde{F}(A', p', c, \xi, \overline{G}, \overline{\Gamma}^k) = \mathcal{H}(B'^t A' B', B'^t p', c, \xi),$$

and using (3.15), (3.20), we have formula (3.9). Note the role of B' which makes $B'^t A' B' : (T_\xi\mathcal{N}, I) \rightarrow (T_\xi^*\mathcal{N}, I)$ symmetric. In particular, if we take ψ as a geodesic

coordinate system around ξ , and $\bar{\psi}$ to be a Euclidean diagonal rotation R with respect to ψ (both are R -related) so that $B' = R$, $\bar{G} = R^t I R = I$, and $\bar{\Gamma}^k = 0$ at the point ξ , then (3.16) can be written as

$$(3.22) \quad \tilde{F}(A, p, c, \xi, I, 0) = \tilde{F}(R^t A R, R^t p, c, \xi, I, 0),$$

that is, as

$$(3.23) \quad \mathcal{H}(A', p', c, \xi) = \mathcal{H}(R^t A' R, R^t p', c, \xi),$$

where $A' : (T_\xi \mathcal{N}, I) \rightarrow (T_\xi^* \mathcal{N}, I)$ is any matrix in $SM_\xi(\mathcal{N}, I)$, $p' \in (T_\xi^* \mathcal{N}, I)$, and R is any Euclidean diagonal rotation in $(T_\xi \mathcal{N}, I)$.

Note that the general expression of rotation invariance is written in terms of \tilde{F} in (3.16), (3.17). \square

Remark 7. Let us write the rotation invariance in the tangent plane in terms of F . If we consider the quadratic form Q in the coordinate system $\psi = (\psi_1, \psi_2) : B_{T_\xi \mathcal{N}}(0, r) \rightarrow \mathcal{N}$ given by

$$Q(v) = \frac{1}{2}(Sv, v) + (p, v) + c,$$

where $S \in SM_\xi(\mathcal{N})$, we consider the diagonal rotation $R : T_\xi \mathcal{N} \rightarrow T_\xi \mathcal{N}$ and define

$$\bar{Q}(w) = \frac{1}{2}(SRv, Rv) + (p, Rv) + c = Q(Rw).$$

Consider the function $C(\zeta) = Q(\psi^{-1}(\zeta))$. In the coordinate system $\bar{\psi} : B_{T_\xi \mathcal{N}}(0, r) \rightarrow \mathcal{N}$ given by $\bar{\psi}(v) = \psi(Rv)$, $C(\zeta) = \bar{Q}(R^{-1}\psi^{-1}(\zeta)) = \bar{Q}(\bar{\psi}^{-1}(\zeta))$. That is, C is expressed by \bar{Q} in the coordinate system $\bar{\psi}$. Both expressions are R -related. Then by the infinitesimal generator axiom

$$\begin{aligned} \frac{T_t(C, \psi)(\xi) - (C, \psi)(\xi)}{t} &\rightarrow F(S, p, \xi, G, \Gamma^k), \\ \frac{T_t(R(C, \psi))(\xi) - R(C, \psi)(\xi)}{t} &\rightarrow F(R^t S R, R^t p, \xi, G, \bar{\Gamma}^k). \end{aligned}$$

Thus

$$(3.24) \quad F(S, p, \xi, G, \Gamma^k) = F(R^t S R, R^t p, \xi, G, \bar{\Gamma}^k).$$

Notice that the metric does not change, but the connection does.

Remark 8. If $BB^t = G^{-1}$, then

$$(3.25) \quad \mathcal{H}(B^t A B, B^t p, c, \xi) = \mathcal{H}(R^t B^t A B R, R^t B^t p, c, \xi),$$

where $A : (T_\xi \mathcal{N}, G) \rightarrow (T_\xi^* \mathcal{N}, G)$ is any matrix in $SM_\xi(\mathcal{N}, G)$, $p \in (T_\xi^* \mathcal{N}, G)$, and R is any Euclidean diagonal rotation in $(T_\xi \mathcal{N}, I)$. Note that $B^t A B : (T_\xi \mathcal{N}, I) \rightarrow (T_\xi^* \mathcal{N}, I)$ is any matrix in $SM_\xi(\mathcal{N}, I)$, and $B^t p \in (T_\xi^* \mathcal{N}, I)$. Note that $BR : (T_\xi \mathcal{N}, I) \rightarrow (T_\xi \mathcal{N}, G)$ satisfies

$$\langle BRv, BRw \rangle_{g^1 \times g^2} = \langle v, w \rangle_{I \times I}.$$

That is, it is an isometry matrix from $(T_\xi \mathcal{N}, I)$ to $(T_\xi \mathcal{N}, G)$.

We can also write the rotation invariance of \mathcal{H} in a different way. Start with (writing $A' = GA - \Gamma(p)$)

$$(3.26) \quad \tilde{F}(A', p, c, \xi, G, \Gamma^k) = \tilde{F}(R^t A' R, R^t p, c, \xi, G, \bar{\Gamma}^k)$$

for any diagonal rotation R in $(T_\xi \mathcal{N}, G)$, and use (3.21) to obtain

$$(3.27) \quad \mathcal{H}(B^t A' B, B^t p, c, \xi) = \mathcal{H}(B^t R^t A' R B, B^t R^t p, c, \xi).$$

Note that RB is a diagonal isometry from $(T_\xi \mathcal{N}, I)$ to $(T_\xi \mathcal{N}, G)$.

Let us comment on the implications of coordinate symmetry. Let

$$(3.28) \quad S = \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}.$$

Let us assume that $\mathcal{M}^1 = \mathcal{M}^2 = \mathcal{M}$ (with different metrics). Let us consider the axiom

$$\begin{aligned} [\text{Axiom of symmetry of the two coordinates}] \quad & \text{If } SC(\xi_1, \xi_2) = C(\xi_2, \xi_1), \text{ then} \\ & T_t(SC) = ST_t(C) \quad \forall t \geq 0, \forall C \in C_b^\infty(\mathcal{M} \times \mathcal{M}). \end{aligned}$$

Note that if the coordinate system around ξ that we use when we compute $T_t(C)$ is $\psi = (\psi_1, \psi_2)$, then we use $S\psi = (\psi_2, \psi_1)$ when we compute $T_t(SC)$. We should write then $T_t(SC, S\psi) = ST_t(C, \psi)$. Perhaps we incur herein a slight abuse of notation.

Since $D_{\mathcal{N}}^2 S(C)(\xi) = SD_{\mathcal{N}}^2 C(S\xi)S$, $D_{\mathcal{N}} S(C)(\xi) = SD_{\mathcal{N}}^2 C(S\xi)$, and, letting $S[\Gamma^{(1)} \otimes \Gamma^{(2)}]^k(\xi) := S(\Gamma^{(1)} \otimes \Gamma^{(2)})^k(S\xi)S$, we have

$$S[\Gamma^{(1)} \otimes \Gamma^{(2)}]^k(\xi) = (\Gamma^{(1)} \otimes \Gamma^{(2)})^k(S\xi),$$

we obtain the following lemma.

LEMMA 3.7. *Let T_t be a multiscale analysis satisfying the axioms, including all architectural axioms, the comparison principle, the gray level Shift invariance, and the symmetry of the two coordinates invariance. Then F satisfies*

$$\begin{aligned} F(SAS, Sp, \xi, SG(\xi), \Gamma^k(S\xi)) &= F(A, p, S\xi, SG(\xi), \Gamma^k(S\xi)) \\ \forall A \in \mathcal{S}(\mathcal{N}), \forall p \in \mathcal{N} \setminus \{0\}, \forall \xi \in \mathcal{N}. \end{aligned}$$

Note that the two last arguments are the same; the invariance is expressed in the first three arguments.

4. The linear case. The proof of the next lemma is elementary and can be found in [5] (particularly in Lemma 5.1).

LEMMA 4.1. *Let \mathcal{M} be a Riemannian manifold. Let D be a matrix such that*

$$RDR^t = D$$

for all rotations R in $(T_\eta \mathcal{M}, G(\eta))$. Then $D = \lambda G(\eta)^{-1}$ for some $\lambda \in \mathbb{R}$.

THEOREM 4.2. *Let T_t be a multiscale analysis on similarity functions satisfying the axioms, including all architectural axioms, the comparison principle, and the gray level shift invariance. Assume that T_t is linear. Then*

$$C_t = F(D_{\mathcal{N}}^2 C, \xi, G),$$

where

$$F(X, \xi, G) = c_{11}(\xi) \text{Tr}((G^1)^{-1}(\xi_1)X_{11}) + 2c_{12}(\xi, G) \text{Tr}(\bar{D}_{12}I^1(\xi_1)^{-1}X_{12}) \\ + c_{22}(\xi) \text{Tr}((G^2)^{-1}(\xi_2)X_{22}),$$

where \bar{D}_{12} is an isometry from $(T_{\xi_1}\mathcal{M}^1, G^1(\xi_1)) \rightarrow (T_{\xi_2}\mathcal{M}^2, G^2(\xi_2))$. The ellipticity of F implies that $c_{11}, c_{22} \geq 0$.

Moreover, we could add that

$$(4.1) \quad 2c_{12}(\xi, G)\bar{D}_{12}I^1(\xi_1)^{-1} = B^2(\xi_2)D'B^1(\xi_1)^t,$$

and the dependence of $c_{12}(\xi, G)\bar{D}_{12}I^1(\xi_1)^{-1}$ on G is only in $B^2(\xi_2)$ (isometry) and $B^1(\xi_1)^t$ (isometry). D' is a matrix that depends only on ξ (see Remark 9 below). We could also write the second term as $2c_{21}(\xi, G)\text{Tr}(\bar{D}_{21}I^2(\xi_2)^{-1}X_{21})$.

Note that the operators $c_{ii}(\xi)\text{Tr}((G^i)^{-1}(\xi_i)X_{ii})$ are multiples of the Laplace–Beltrami operator. Notice also that there are no first order terms in these operators. They cannot couple with vectors so that we have the required invariance induced by the rotations of tangent planes.

Proof. Since T_t is gray level shift invariant, \tilde{F} does not depend on c . On the other hand, it does not depend on Γ^k . The linearity of T_t and Theorem 3.2 imply that in terms of the function \tilde{F}

$$\tilde{F}(rX_1 + sX_2, rp_1 + sp_2, \xi, G) = r\tilde{F}(X_1, p_1, \xi, G) + s\tilde{F}(X_2, p_2, \xi, G)$$

for any $X_1, X_2 \in \text{SM}_\xi(\mathcal{N})$, any $p_1, p_2 \in T_\xi^*\mathcal{N}$, and any $r, s \in \mathbb{R}$. By taking $X_1 = X$, $X_2 = 0$, $p_1 = 0$, $p_2 = p$, $r = 1$, $s = 1$, we write

$$\tilde{F}(X, p, \xi, G) = \tilde{F}(X, 0, \xi, G) + \tilde{F}(0, p, \xi, G) =: K'(X, \xi, G) + K''(p, \xi, G),$$

where K' is linear in X and K'' is linear in p . Moreover, from the rotation invariance of \tilde{F} ,

$$\tilde{F}(X, p, \xi, G) = \tilde{F}(R^tXR, R^tp, \xi, G) \quad \forall \text{ diagonal rotations } R \text{ in } (T_\xi\mathcal{N}, G(\xi))$$

(diagonal means $R = (R_1, R_2)$ and R_1, R_2 are related), we deduce that

$$(4.2) \quad K'(X, \xi, G) = K'(R^tXR, \xi, G),$$

$$(4.3) \quad K''(p, \xi, G) = K''(R^tp, \xi, G).$$

Let us write

$$X = \begin{pmatrix} X_{11} & X_{12} \\ X_{21} & X_{22} \end{pmatrix},$$

where $X_{21} = X_{12}^t$. We also write

$$R = \begin{pmatrix} R_1 & 0 \\ 0 & R_2 \end{pmatrix},$$

where $R_1 : (T_{\xi_1}\mathcal{M}^1, G^1(\xi_1)) \rightarrow (T_{\xi_1}\mathcal{M}^1, G^1(\xi_1))$ and $R_2 : (T_{\xi_2}\mathcal{M}^2, G^2(\xi_2)) \rightarrow (T_{\xi_1}\mathcal{M}^2, G^2(\xi_2))$, with $R_2\bar{P}(\xi) = P(\xi)R_1$. Then

$$R^tXR = \begin{pmatrix} R_1^tX_{11}R_1 & R_1^tX_{12}R_2 \\ R_2^tX_{21}R_1 & R_2^tX_{22}R_2 \end{pmatrix}.$$

Since K' is linear in X , we have

$$K'(X, \xi, G) = K'_{11}(X_{11}, \xi, G) + K'_{12}(X_{12}, \xi, G) + K'_{21}(X_{21}, \xi, G) + K'_{22}(X_{22}, \xi, G),$$

where each $K'_{ij}(X_{ij}, \xi, G)$ is linear in X_{ij} and by (4.2) we have

$$(4.4) \quad \begin{aligned} K'_{11}(R_1^t X_{11} R_1, \xi, G) &= K'_{11}(X_{11}, \xi, G), \\ K'_{12}(R_1^t X_{12} R_2, \xi, G) + K'_{21}(R_2^t X_{21} R_1, \xi, G) &= K'_{12}(X_{12}, \xi, G) + K'_{21}(X_{21}, \xi, G), \\ K'_{22}(R_2^t X_{22} R_2, \xi, G) &= K'_{22}(X_{22}, \xi, G). \end{aligned}$$

At ξ fixed, and for $ij = 11, 22$, $K'_{ij}(X_{ij}, \xi, G)$ is a symmetric linear function of the eigenvalues of X_{ij} . Then there exists a matrix $D_{ii} : (T_{\xi_i} \mathcal{M}^i, G^i(\xi_i)) \rightarrow (T_{\xi_i} \mathcal{M}^i, G^i(\xi_i))$ (depending on ξ, G) such that

$$K'_{ij}(X_{ij}, \xi, G) = \text{Trace}(D_{ij} X_{ij}).$$

From the rotation invariance

$$\text{Trace}(D_{ij} X_{ij}) = \text{Trace}(D_{ij} R_i^t X_{ij} R_j) = \text{Trace}(R_j D_{ij} R_i^t X_{ij}).$$

Since this is true for all X_{ij} , then $R_j D_{ij} R_i^t = D_{ij}$. By Lemma 4.1 we have that $D_{ij} = c_{ij}(\xi, G)(G^i)^{-1}(\xi_i)$ for some constant $c_{ij}(\xi, G)$.

Now, $K'_{12}(X_{12}, \xi, G) + K'_{21}(X_{21}, \xi, G)$ is a linear function of X_{12} . Thus there is a matrix D'_{ij} such that

$$K'_{12}(X_{12}, \xi, G) + K'_{21}(X_{21}, \xi, G) = \text{Trace}(D'_{12} X_{12}).$$

Since the map has to be an endomorphism, $D'_{12} X_{12}$ has to be a map $(T_{\xi_2} \mathcal{M}^2, G^2(\xi_2)) \rightarrow (T_{\xi_2} \mathcal{M}^2, G^2(\xi_2))$. Since $X_{12} : (T_{\xi_2} \mathcal{M}^2, G^2(\xi_2)) \rightarrow (T_{\xi_1} \mathcal{M}^1, G^1(\xi_1))^*$, we write the map $D'_{12} = D_{12} I(\xi_1)^{-1}$, where $I(\xi_1) : (T_{\xi_1} \mathcal{M}^1, G^1(\xi_1)) \rightarrow (T_{\xi_1} \mathcal{M}^1, G^1(\xi_1))^*$ and $D_{12} : (T_{\xi_1} \mathcal{M}^1, G^1(\xi_1)) \rightarrow (T_{\xi_2} \mathcal{M}^2, G^2(\xi_2))$. Using the rotation invariance (4.4), we have

$$(4.5) \quad \begin{aligned} \text{Trace}(D_{12} I^1(\xi_1)^{-1} X_{12}) &= \text{Trace}(D_{12} I^1(\xi_1)^{-1} R_1^t X_{12} R_2) \\ &= \text{Trace}(R_2 D_{12} I^1(\xi_1)^{-1} R_1^t X_{12}). \end{aligned}$$

Note that all maps inside the traces map $(T_{\xi_2} \mathcal{M}^2, G^2(\xi_2)) \rightarrow (T_{\xi_2} \mathcal{M}^2, G^2(\xi_2))$. This implies that

$$D_{12} I(\xi_1)^{-1} = R_2 D_{12} I(\xi_1)^{-1} R_1^t,$$

that is,

$$D_{12} = R_2 D_{12} I^1(\xi_1)^{-1} R_1^t I^1(\xi_1),$$

as maps from $(T_{\xi_1} \mathcal{M}^1, G^1(\xi_1)) \rightarrow (T_{\xi_2} \mathcal{M}^2, G^2(\xi_2))$. Let us observe that $I^1(\xi_1)^{-1} R_1^t I^1(\xi_1) : (T_{\xi_1} \mathcal{M}^1, G^1(\xi_1)) \rightarrow (T_{\xi_1} \mathcal{M}^1, G^1(\xi_1))$ is an isometry and denote it by $\overline{R}_1^t = I^1(\xi_1)^{-1} R_1^t I^1(\xi_1)$. We can write

$$(4.6) \quad D_{12} \overline{R}_1^{-t} = R_2 D_{12}.$$

Let us interpret R_2 as a representation of the isometry group of $(T_{\xi_2}\mathcal{M}^2, G^2(\xi_2))$, and \bar{R}_1^{-t} is a representation of the isometry group of $(T_{\xi_1}\mathcal{M}^1, G^1(\xi_1))$. Denote them by $\bar{\rho}(\mathcal{R})$ and $\rho(\mathcal{R})$, respectively (\mathcal{R} represents a rotation). Note the slight abuse of notation writing \mathcal{R} in both cases, but note that R_1 is determined by R_2 , since both are $P(\xi)$ -related. Then $\rho(I) = I$ and $\bar{\rho}(I) = I$. Note also that $I = \rho(\mathcal{R}\mathcal{R}^{-1}) = \rho(\mathcal{R})\rho(\mathcal{R}^{-1})$; thus $\rho(\mathcal{R}^{-1}) = \rho(\mathcal{R})^t$ (they are isometries). The same is true for $\bar{\rho}(\mathcal{R})$. We rewrite (4.6) as

$$(4.7) \quad D_{12}\rho(\mathcal{R}) = \bar{\rho}(\mathcal{R})D_{12} \quad \forall \mathcal{R}.$$

By transposing we have

$$(4.8) \quad \rho(\mathcal{R})^t D_{12}^t = D_{12}^t \bar{\rho}(\mathcal{R})^t \quad \forall \mathcal{R}.$$

After multiplying by D_{12} ,

$$(4.9) \quad D_{12}\rho(\mathcal{R})^t D_{12}^t = D_{12}D_{12}^t \bar{\rho}(\mathcal{R})^t \quad \forall \mathcal{R}.$$

Writing (4.7) with \mathcal{R}^{-1} instead of \mathcal{R} and using that $\rho(\mathcal{R}^{-1}) = \rho(\mathcal{R})^t$, $\bar{\rho}(\mathcal{R}^{-1}) = \bar{\rho}(\mathcal{R})^t$, we have

$$(4.10) \quad D_{12}\rho(\mathcal{R})^t = \bar{\rho}(\mathcal{R})^t D_{12} \quad \forall \mathcal{R}.$$

Combining (4.10) with (4.9), we have

$$(4.11) \quad \bar{\rho}(\mathcal{R})^t D_{12}D_{12}^t = D_{12}D_{12}^t \bar{\rho}(\mathcal{R})^t \quad \forall \mathcal{R}.$$

Then, by Schur's lemma [19], there is a constant $c \in \mathbb{R}$ such that $D_{12}D_{12}^t = cI$. Thus, either $D_{12} = 0$ or $\frac{1}{\sqrt{c}}D_{12}$ is an isometry in $(T_{\xi_1}\mathcal{M}^1, G^1(\xi_1)) \rightarrow (T_{\xi_2}\mathcal{M}^2, G^2(\xi_2))$. Note that this cannot be improved since, reading this backward, we have that (4.4) holds. Let us note the constant c as $2c_{12}(\xi, G)$.

We have proved that

$$(4.12) \quad K'_{12}(X_{12}, \xi, G) + K'_{21}(X_{21}, \xi, G) = 2c_{12}(\xi, G)\text{Trace}(\bar{D}_{12}I^1(\xi_1)^{-1}X_{12}),$$

where $\bar{D}_{12} : (T_{\xi_1}\mathcal{M}^1, G^1(\xi_1)) \rightarrow (T_{\xi_2}\mathcal{M}^2, G^2(\xi_2))$ is an isometry.

Since for $i = 1, 2$, we can write

$$\begin{aligned} K'_{ii}(X_{ii}, \xi, G) &= c_{ii}(\xi, G)\text{Trace}((G^i)^{-1}(\xi_i)X_{ii}) = c_{ii}(\xi, G)\text{Trace}(B^i(\xi_i)(B^i)^t(\xi_i)X_{ii}) \\ &= c_{ii}(\xi, G)\text{Trace}((B^i)^t(\xi_i)X_{ii}B^i(\xi_i)) = \mathcal{H}_{ii}(B^i(\xi_i)^t X_{ii} B^i(\xi_i), 0, \xi), \end{aligned}$$

where \mathcal{H}_{ii} is linear in its first argument (see (3.21)), we deduce that $c_{ii}(\xi, G)$ does not depend on G . The ellipticity of \tilde{F} proves that $c_{ii}(\xi) \geq 0$, $i = 1, 2$.

Let us prove that $K''(p, \xi, G) = 0$. Now, by (4.3) we have

$$K''(p, \xi, G) = K''(R^t p, \xi, G)$$

for all diagonal rotations as above. Let $p = (p_1, p_2)$. Then we may write

$$K''(p, \xi, G) = K''_1(p_1, \xi, G) + K''_2(p_2, \xi, G),$$

where K''_i is linear in p_i (ξ, G fixed). Thus, letting $p_1 = 0$ and $p_2 = 0$, respectively, we deduce

$$K''_i(p_i, \xi, G) = K''_i(R_i^t p_i, \xi, G)$$

for $i = 1, 2$. Thus K_i'' does not depend on p_i but depends only on its modulus; that is,

$$K_i''(p_i, \xi, G) = \bar{K}_i''(|p_i|_{(g^i)^{-1}}, \xi, G)$$

for some function \bar{K}_i'' .

Let us compute the modulus. Observe that

$$\langle R_i^t p_i, R_i^t p_i' \rangle = ((G^i)^{-1}(\xi_i) R_i^t p_i, R_i^t p_i') = (R_i (G^i)^{-1}(\xi_i) R_i^t p_i, p_i').$$

From $R_i^t G^i(\xi_i) R_i = G^i(\xi_i)$, we have $R_i (G^i)^{-1}(\xi_i) R_i^t = (G^i)^{-1}(\xi_i)$. Thus

$$(R_i (G^i)^{-1}(\xi_i) R_i^t p_i, p_i') = ((G^i)^{-1}(\xi_i) p_i, p_i') = \langle p_i, p_i' \rangle.$$

Thus $|R_i^t p_i|_{(g^i)^{-1}(\xi_i)} = |p_i|_{(g^i)^{-1}(\xi_i)}$ for any covector p_i . Then

$$2\bar{K}_i''(|p_i|_{(g^i)^{-1}(\xi_i)}, \xi, G) = K_i''(p_i, \xi, G) + K_i''(-p_i, \xi, G) = K_i''(0, \xi, G) = 0.$$

Our claim is proved. \square

Remark 9. In the context of the above proof, let us analyze the dependence of $2c_{12}(\xi, G)\bar{D}_{12}I^1(\xi_1)^{-1}$ on $G(\xi)$. From

$$K'_{12}(X_{12}, \xi, G) = \mathcal{H}_{12}(B^1(\xi_1)^t X_{12} B^2(\xi_2), 0, \xi),$$

where \mathcal{H}_{12} is linear in its first argument (see (3.21)), we have

$$2c_{12}(\xi, G)\text{Trace}(\bar{D}_{12}I^1(\xi_1)^{-1}X_{12}) = \text{Trace}(D'B^1(\xi_1)^t X_{12} B^2(\xi_2)),$$

where $D' : (T_{\xi_1}\mathcal{M}^1, \bar{G}^1(\xi_1))^* \rightarrow (T_{\xi_2}\mathcal{M}^2, \bar{G}^2(\xi_2))$ depends only on ξ , and $B^i(\xi_i) = B_{G^i\bar{G}^i}(\xi_i) : (T_{\xi_i}\mathcal{M}^i, \bar{G}^i(\xi_i)) \rightarrow (T_{\xi_i}\mathcal{M}^i, G^i(\xi_i))$, $X_{12} : (T_{\xi_2}\mathcal{M}^2, \bar{G}^2(\xi_2)) \rightarrow (T_{\xi_1}\mathcal{M}^1, \bar{G}^1(\xi_1))^*$. Let us write

$$\bar{D}_{12}I^1(\xi_1)^{-1} = B^2(\xi_2)\bar{D}'_{12}B^1(\xi_1)^t.$$

Then

$$\begin{aligned} \text{Trace}(D'B^1(\xi_1)^t X_{12} B^2(\xi_2)) &= 2c_{12}(\xi, G)\text{Trace}(B^2(\xi_2)\bar{D}'_{12}B^1(\xi_1)^t X_{12}) \\ &= 2c_{12}(\xi, G)\text{Trace}(\bar{D}'_{12}B^1(\xi_1)^t X_{12} B^2(\xi_2)). \end{aligned}$$

Thus

$$2c_{12}(\xi, G)\bar{D}'_{12} = D'.$$

Multiplying by $B^2(\xi_2)$ to the left and by $B^1(\xi_1)^t$ to the right, we have

$$2c_{12}(\xi, G)\bar{D}_{12}I^1(\xi_1)^{-1} = B^2(\xi_2)D'B^1(\xi_1)^t.$$

The dependence of $c_{12}(\xi, G)\bar{D}_{12}I^1(\xi_1)^{-1}$ on G is only in $B^2(\xi_2)$ (isometry) and $B^1(\xi_1)^t$ (isometry).

If, in addition to the assumptions of Theorem 4.2, we assume that T_t satisfies the axiom of symmetry of the two coordinates, we have $c_{11}(S\xi) = c_{22}(\xi)$. Concerning the second term, we exploit the expression (4.1), and the axiom of symmetry of the two coordinates implies that

$$\text{Trace}(B^2(\xi_2)D'(\xi)B^1(\xi_1)^t X_{12}) = \text{Trace}(B^1(\xi_1)D'(S\xi)B^2(\xi_2)^t X_{21})$$

for all X_{12} . By transposing in the last expression we can continue the equalities

$$= \text{Trace}(X_{12}B^2(\xi_2)D'(S\xi)^t B^1(\xi_1)^t) = \text{Trace}(B^2(\xi_2)D'(S\xi)^t B^1(\xi_1)^t X_{12}).$$

Since this holds for any X_{12} , we have that

$$B^2(\xi_2)D'(\xi)B^1(\xi_1)^t = B^2(\xi_2)D'(S\xi)^t B^1(\xi_1)^t.$$

This implies that

$$D'(S\xi) = D'(\xi)^t.$$

4.1. The case of $(\mathcal{M}^r, \mathbf{g}^r(\mathbf{x})) = (\mathbb{R}^N, \mathbf{g}^r(\mathbf{x}))$. To fix ideas we consider $\mathcal{M}^1 = \mathcal{M}^2 = \mathcal{M} = \mathbb{R}^N$ and $g_{ij}^r(x)$ to be general metrics in \mathbb{R}^N , $r = 1, 2$. We know that $e_i = G^r(x)^{-1/2}f_i$ is an orthonormal basis of $(T_x\mathcal{M}^r, g^r(x))$ if f_i is a Euclidean orthonormal basis. Let $I^r(x) : (\mathbb{R}^N, g^r(x)) \rightarrow (\mathbb{R}^N, (g^r)^{-1}(x))$ be given by $I^r(x)e_i = e_i^*$. Then

$$I^r(x) = G^r(x).$$

If $B^r(x)$ satisfies $B^r(x)I^r(x)^{-1}B^r(x)^t = G^r(x)^{-1}$, then we can take $B^r(x) = I$.

We can define $P(x, y)(v) = G^2(y)^{-1/2}G^1(x)^{1/2}v$, $v \in \mathbb{R}^N$, as the a priori connection of x and y . Then $|P(x, y)v|_{g^2} = |v|_{g^1}$ for all $(x, y) \in \mathbb{R}^{2N}$. Recall that $\bar{D}_{1,2} : (\mathbb{R}^N, g^1(x)) \rightarrow (\mathbb{R}^N, g^2(y))$ is an isometry, in this case given by $\bar{D}_{1,2} = G^2(y)^{-1/2}G^1(x)^{1/2}$. Then (4.1) is

$$2c_{12}(x, y)\bar{D}_{1,2}I^1(x)^{-1} = 2c_{12}(x, y)G^2(y)^{-1/2}G^1(x)^{-1/2}.$$

The PDE obtained is

$$(4.13) \quad C_t = a(x, y)\Delta_{\mathcal{M}_x}C + 2c_{12}(x, y)\text{Tr}(G^2(y)^{-1/2}G^1(x)^{-1/2}D_{xy}C) + c(x, y)\Delta_{\mathcal{M}_y}C,$$

where

$$\Delta_{\mathcal{M}_x}C = \text{Tr}(G^1(x)^{-1}(D_{xx}u(x) - \Gamma^{(1)}(Du)(x))).$$

The same is true for the operator $\Delta_{\mathcal{M}_y}$.

Remark 10. Note that (first by transposition and then by reordering) we have

$$\begin{aligned} \text{Tr}(G^2(y)^{-1/2}G^1(x)^{-1/2}D_{xy}C) &= \text{Tr}(D_{yx}CG^1(x)^{-1/2}G^2(y)^{-1/2}) \\ &= \text{Tr}(G^1(x)^{-1/2}G^2(y)^{-1/2}D_{yx}C), \end{aligned}$$

which is a symmetric expression in (x, y) . If T_t is symmetric in (x, y) , then c_{12} is also symmetric.

In the symmetric case, the matrix associated to the operator (4.13) is

$$\begin{pmatrix} a(x, y)G^1(x)^{-1} & c_{12}(x, y)G^2(y)^{-1/2}G^1(x)^{-1/2} \\ c_{12}(x, y)G^1(x)^{-1/2}G^2(y)^{-1/2} & c(x, y)G^2(y)^{-1} \end{pmatrix}.$$

It is positive semidefinite if and only if $a, c \geq 0$ and $ac - c_{12}^2 \geq 0$.

Remark 11. This will permit us to also construct an operator in the case of video. In that case, $N = 3$ and $\mathcal{M} = \{(t, x) : t \in \mathbb{R}, x \in \mathbb{R}^2\}$. Let us consider the metric $g(t, x)_{ij}$ so that if (s, y) denote the coordinates in $T_{(t,x)}\mathcal{M}$,

$$(4.14) \quad g(x, t)_{ij}dx_i dx_j = A(t, x)(y - v(x, t)s)^2 + B(t, x)s^2.$$

As an example, we can take $A(t, x) = \alpha + |\nabla_x I|^2$, $B(t, x) = \beta + (\partial_v I)^2$, $\alpha, \beta > 0$.

4.2. The case of conformal metrics in \mathbb{R}^N . To fix ideas take $\mathcal{M}^1 = \mathcal{M}^2 = \mathcal{M} = \mathbb{R}^N$ and $g_{ij}^r(x) = \lambda^r(x)^2 \delta_{ij}$, $\lambda^r(x) > 0$ for $x \in \mathbb{R}$. We can also consider $\mathcal{M} = \mathbb{T}^N$, where \mathbb{T} represents the circle, assuming that we can deploy functions on \mathbb{T} to \mathbb{R}^N by parity and periodic extension. In that case, $\lambda^r(x)$ is similarly extended. Note the connection

$$\Gamma_{ij}^{(r)k}(x) = \frac{\partial}{\partial x^i} \ln \lambda^r \delta_{jk} + \frac{\partial}{\partial x^j} \ln \lambda^r \delta_{ik} - \frac{\partial}{\partial x^k} \ln \lambda^r \delta_{ij}.$$

We also have for $u(x)$

$$D_{\mathcal{M}x}^2 u(x) = D_{xx} u(x) - \Gamma^{(r)}(Du)(x),$$

where

$$\Gamma^{(r)}(Du) := (\Gamma_{ij}^{(r)k}(x))_{ij}(\cdot_k) Du(x) = Du \otimes D \ln \lambda^r + D \ln \lambda^r \otimes Du - D \ln \lambda^r \cdot Du I,$$

where we have denoted by (\cdot_k) the contraction (like scalar product) in the variable k with the coordinates of Du . Thus

$$\text{Tr}(\Gamma^{(r)}(Du)) = -(N-2)D \ln \lambda^r \cdot Du.$$

Note that $e_i = \frac{1}{\lambda^r(x)} f_i$ is an orthonormal basis of $(\mathbb{R}^N, \lambda^r(x))$, when f_i is a Euclidean orthonormal basis of \mathbb{R}^N . Then $e_i^* = \lambda^r(x) f_i$ is the dual basis. Then the operator $I^r(x) : (\mathbb{R}^N, g^r(x)) \rightarrow (\mathbb{R}^N, (g^r)^{-1}(x))$ such that $I^r(x) e_i = e_i^*$ is given by $I^r(x) = \lambda^r(x)^2 I$. If $B^r(x)$ satisfies $B^r(x) I^r(x)^{-1} B^r(x)^t = G^r(x)^{-1}$, then we may take $B^r(x) = I$.

We define $P(x, y)(v) = \frac{\lambda^1(x)}{\lambda^2(y)} v$, $v \in \mathbb{R}^N$, as the a priori connection of x and y . Then $|P(x, y)v|_{g^2} = |v|_{g^1}$ for all $(x, y) \in \mathbb{R}^{2N}$.

Note that

$$\begin{aligned} \text{Tr}_{g^1}(D_{\mathcal{M}xx}^2 u(x)) &= \text{Tr}((G^1)(x)^{-1} D_{\mathcal{M}xx}^2 u(x)) = \frac{1}{\lambda^1(x)^2} \text{Tr}(D_{xx} u(x) - \Gamma^{(1)}(Du)(x)) \\ &= \frac{1}{\lambda^1(x)^2} (\Delta u(x) + (N-2)D \ln \lambda^1 \cdot Du) = \frac{1}{(\lambda^1(x))^N} \text{div}((\lambda^1(x))^{N-2} Du(x)) \\ &= \frac{1}{\sqrt{\det(g^1(x))}} \text{div}\left(\sqrt{\det(g^1(x))} (g^1(x))^{-1} Du(x)\right) = \Delta_{\mathcal{M}x} u(x), \end{aligned}$$

which is the Laplace–Beltrami operator.

Let us write (4.1) as

$$2c_{12}(x, y) \bar{D}_{1,2} I^1(x)^{-1} = B^2(y) D'(x, y) B^1(x),$$

where $\bar{D}_{1,2} : (\mathbb{R}^N, (\lambda^1(x))^2 I) \rightarrow (\mathbb{R}^N, (\lambda^2(y))^2 I)$ is an isometry, in this case given by $\bar{D}_{1,2} = \frac{\lambda^1(x)}{\lambda^2(y)} I$. Then we have

$$2c_{12}(x, y) \frac{\lambda^1(x)}{\lambda^2(y)} \frac{1}{(\lambda^1(x))^2} I = D'(x, y).$$

That is,

$$D'(x, y) = 2 \frac{c_{12}(x, y)}{\lambda^1(x) \lambda^2(y)} I.$$

Thus, the linear operator on $C(t, x, y)$ can be written as

$$(4.15) \quad C_t = \frac{a(x,y)}{(\lambda^1(x))^2} (\Delta_x C + (N-2)D_x \ln \lambda^1(x) \cdot D_x C) + 2 \frac{c_{12}(x,y)}{\lambda^1(x)\lambda^2(y)} \text{Tr}(D_{xy} C) \\ + \frac{c(x,y)}{(\lambda^2(y))^2} (\Delta_y C(y) + (N-2)D_y \ln \lambda^2(y) \cdot D_y C)$$

or

$$(4.16) \quad C_t = a(x,y)\Delta_{\mathcal{M}_x} C + 2 \frac{c_{12}(x,y)}{\lambda^1(x)\lambda^2(y)} \text{Tr}(D_{xy} C) + c(x,y)\Delta_{\mathcal{M}_y} C$$

for functions $a(x, y), c_{12}(x, y), c(x, y)$ so that the operator is elliptic (that is, if and only if $a|v_1|^2 + 2c_{12} < v_1, v_2 > + c|v_2|^2 \geq 0$ for all v_1, v_2). This is the case if and only if $a, c \geq 0$ and $ac - c_{12}^2 \geq 0$. Indeed, if the operator is elliptic, then by writing $v_1 = \alpha e, v_2 = \beta e$ we get $a\alpha^2 + 2c_{12}\alpha\beta + c\beta^2 \geq 0$ for all α, β ; thus $ac - c_{12}^2 \geq 0$. If $ac - c_{12}^2 \geq 0$, then $a|v_1|^2 + 2c_{12} < v_1, v_2 > + c|v_2|^2 \geq a|v_1|^2 - 2|c_{12}||v_1||v_2| + c|v_2|^2 \geq 0$. If T_t is symmetric in (x, y) , then c_{12} is also symmetric.

If $N = 1$, we simply have

$$\Gamma^r(x) = \frac{(\lambda^r)'(x)}{\lambda^r(x)} = (\ln \lambda^r(x))',$$

and (4.16) is

$$(4.17) \quad C_t = \frac{a}{\lambda^1(x)^2} (C_{xx} - (\ln \lambda^1(x))' C_x) + 2 \frac{c_{12}(x,y)}{\lambda^1(x)\lambda^2(y)} C_{xy} + \frac{c(x,y)}{(\lambda^2(y))^2} (C_{yy} - (\ln \lambda^2(y))' C_y).$$

4.3. The case where $(\mathcal{M}^r, g) = (\mathbb{R}^N, I)$. Let us start by considering a general metric g in \mathbb{R}^N , and $\overline{P}(\xi)(v) = v$ for all $v \in \mathbb{R}^N$. Then (2.7) is saying that the metric g is constant and $\overline{P}(\xi)$ is also the identity. If $P(\xi)(v)$ is any a priori connection map, we have the result stated in Theorem 4.2.

Let us consider the case $(\mathcal{M}^r, g) = (\mathbb{R}^N, I)$. We denote $\xi = (x, y) \in \mathbb{R}^N \times \mathbb{R}^N$. We do not subsume this under the general manifold case, because we can assume a different set of invariances that permits the operator to have first order terms. Let us consider translation and rotation invariance.

$$[\text{Translation invariance}] T_t(\tau_{a,b} C) = \tau_{a,b} T_t C \quad \forall t \geq 0, \forall C \in C_b^\infty(\mathbb{R}^N \times \mathbb{R}^N), \forall a, b \in \mathbb{R}^N,$$

$$\text{where } \tau_{a,b} C(x, y) = C(x + a, y + b).$$

$$[\text{Rotation invariance}] T_t(RC) = RT_t C \quad \forall t \geq 0, \forall C \in C_b^\infty(\mathbb{R}^N \times \mathbb{R}^N), \forall R \in O(N),$$

where $RC(x, y) = C(Rx, Ry)$. We have denoted by $O(N)$ the rotations in \mathbb{R}^N .

THEOREM 4.3. *Let T_t be a multiscale analysis in (\mathbb{R}^N, I) satisfying the axioms, including all architectural axioms, the comparison principle, the gray level shift invariance, and rotation invariance. Assume that T_t is linear. Then*

$$(4.18) \quad C_t = F(D^2 C, DC, z),$$

where

$$F(A, v, z) = \sum_{i,j=1}^2 c_{ij}(z) \text{Tr} A_{ij} + \langle b(z), p \rangle$$

for some functions $c_{ij}(z) \in \mathbb{R}$, $b(z) \in \mathbb{R}^{2N}$, $i, j = 1, 2$, such that $c_{ij}(R \circ z) = c_{ij}(z)$, $b(R \circ z) = R \circ b(z)$ for all $R \in O(N)$ and all $z \in \mathbb{R}^{2N}$. The ellipticity of F implies that $(c_{ij}(z))$ is a positive definite matrix for all $z \in \mathbb{R}^{2N}$.

Moreover, if we assume that T_t is translation invariant, then

$$F(A, v, z) = \sum_{i,j=1}^2 c_{ij} \text{Tr} A_{ij},$$

where c_{ij} are constants.

Note the difference between this and the statement of Theorem 4.2; the difference is due to the assumption of rotation invariance that involves the action of the rotation on (x, y) .

Thus the PDE is

$$(4.19) \quad C_t = c_{11}(z) \Delta_x C + 2c_{12}(z) \text{Tr}(D_{xy}^2 C) + c_{22}(z) \Delta_y C \\ + \langle b_1(z), D_x C \rangle + \langle b_2(z), D_y C \rangle.$$

An example is

$$(4.20) \quad C_t = \Delta_x C + 2 \text{Tr}(D_{xy}^2 C) + \Delta_y C.$$

Let $C(t, x, y) = \int_{\mathbb{R}^N} g_t(z) C(0, x+h, y+h) dh$, where g_t is the Gaussian of scale t . Then $C(t, x, y)$ is a solution of (4.20) with initial condition $C(0, x, y)$. If $C(0, x, y) = I(x)J(y)$, then $C(t, x, y) := \int_{\mathbb{R}^N} g_t(z) I(x+h)J(y+h) dh$. If $C(0, x, y) = (I(x) - J(y))^2$, then $C(t, x, y) := \int_{\mathbb{R}^N} g_t(z) (I(x+h) - J(y+h))^2 dh$. Another example is $C(0, x, y) = \sum_{i=1}^N Z_i(x)Z_i(y)$, where $Z(x) = (Z_i(x))_{i=1}^N$ is the direction of the gradient of I .

Proof. Observe that if $L \in GL(N)$ and $C_L(x, y) = C(Lx, Ly)$, then

$$DC_L(x, y) = \begin{pmatrix} L^t D_x C(Lx, Ly) \\ L^t D_y C(Lx, Ly) \end{pmatrix}, \\ D^2 C_L(x, y) = \begin{pmatrix} L^t D_{xx} C(Lx, Ly) L & L^t D_{xy} C(Lx, Ly) L \\ L^t D_{yx} C(Lx, Ly) L & L^t D_{yy} C(Lx, Ly) L \end{pmatrix}.$$

To simplify the notation we write $DC_L(x, y) = L^t \circ DC(Lx, Ly)$ and $D^2 C_L(x, y) = L^t \circ D^2 C(Lx, Ly) \circ L$.

Let $z = (x, y)$. By the axioms above, $F = F(A, p, z)$. As above, from the linearity of T_t we can write

$$F(A, p, z) =: F'(A, z) + F''(p, z),$$

where F' is linear in A and F'' is linear in p . Moreover, if we assume that T_t is translation invariant, then F', F'' do not depend on z .

Then the rotation invariance axiom implies that

$$F(R^t \circ A \circ R, R^t \circ p, z) = F(A, p, R \circ z)$$

for all $R \in O(N)$, $A \in \mathcal{S}(2N)$, $p \in \mathbb{R}^{2N}$, $x, y \in \mathbb{R}^N$. Thus

$$F'(R^t \circ A \circ R, R^t \circ z) + F''(R^t \circ p, R^t \circ z) = F'(A, z) + F''(p, z).$$

Now, since F' and F'' are linear in their first arguments, we have $F'(0, z) = 0$, $F''(0, z) = 0$. Thus

$$(4.21) \quad F'(R^t \circ A \circ R, R^t \circ z) = F'(A, z),$$

$$(4.22) \quad F''(R^t \circ p, R^t \circ z) = F''(p, z)$$

for all values of their arguments. Let us write

$$A = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix},$$

where $A_{21} = A_{12}$. By linearity of F' in its first argument we have

$$F'(A, z) = F'_{11}(A_{11}, z) + 2F'_{12}(A_{12}, z) + F'_{22}(A_{22}, z),$$

where each $F'_{ij}(A_{ij}, z)$ is linear in A_{ij} and $F'_{ij}(R^t A_{ij} R, R^t z) = F'_{ij}(A_{ij}, z)$. At z fixed, $F'_{ij}(A_{ij}, z)$ is a symmetric linear function of the eigenvalues of A_{ij} , that is, a linear function of $\text{Tr}(A_{ij})$. That is, there exists $c_{ij}(z)$ such that

$$F'_{ij}(A_{ij}, z) = c_{ij}(z) \text{Tr}(A_{ij}).$$

Moreover, $c_{ij}(R^t \circ z) = c_{ij}(z)$.

Similarly, for z fixed, $F''(p, z)$ is a linear function of p ; that is, there are some coefficients $b(z) = (b_1(z), b_2(z)) \in \mathbb{R}^{2N}$ such that $F''(p, z) = \langle b(z), p \rangle$. By (4.22) we have

$$\langle b(R \circ z), R \circ p \rangle = \langle b(z), p \rangle \quad \forall p \in \mathbb{R}^{2N}, \forall R \in O(N)$$

and

$$(4.23) \quad \langle b_i(R \circ z), R p_i \rangle = \langle b_i(z), p_i \rangle \quad \forall p_i \in \mathbb{R}^N, \forall R \in O(N), \forall i = 1, 2.$$

This implies that

$$(4.24) \quad R^t b_i(R \circ z) = b_i(z) \quad \forall R \in O(N), \forall i = 1, 2.$$

If we assume that T_t is translation invariant, then $c_{ij}(z) = c_{ij}$ are constants and $b(z)$ does not depend on z . Then, from (4.24) we deduce that $b(z) = 0$. Then

$$F(A, p, z) = \sum_{i,j=1}^2 c_{ij} \text{Tr} A_{ij}. \quad \square$$

Remark 12. Let us give some examples of functions $b(z)$. We can take L_1, L_2 so that $L_i(R \circ z) = L_i(z)$ for all $R \in O_z(N)$. Then

$$b(z) = \begin{pmatrix} L_1(z)x \\ L_2(z)y \end{pmatrix}$$

and

$$b(z) = \begin{pmatrix} L_1(z)(x - y) \\ L_2(z)(x - y) \end{pmatrix}$$

satisfy (4.24).

If, in addition to the assumptions of Theorem 4.3, we assume that T_t satisfies the axiom of symmetry of the two coordinates, we have $c_{11}(Sz) = c_{22}(z)$, $c_{12}(Sz) = c_{12}(z)$ and $b_1(Sz) = b_2(z)$ for all $z \in \mathbb{R}^{2N}$.

If we assume that T_t satisfies the scale invariance axiom,

[Scale invariance] For any $\lambda > 0$ and $t \geq 0$ there exists $t' \geq 0$ such that
 $T_t(D_\lambda C) = D_\lambda T_{t'} C \quad \forall C \in C_b^\infty(\mathbb{R}^N \times \mathbb{R}^N)$, where $D_\lambda C(x, y) = C(\lambda x, \lambda y)$,

and $t \rightarrow T_t$ is one-to-one, then using the arguments in [1, section 6, Lemma 1] or in [16, Chapter 20, Lemma 20.20], after a suitable time rescaling we have

$$F(\lambda^2 A, \lambda p, z) = \lambda^2 F(A, p, \lambda z).$$

Then

$$\begin{aligned} & \lambda^2 (c_{11}(z)\text{Tr}(A_{11}) + 2c_{12}(z)\text{Tr}(A_{12}) + c_{22}(z)\text{Tr}(A_{22})) + \lambda \langle b(z), p \rangle \\ &= \lambda^2 (c_{11}(\lambda z)\text{Tr}(A_{11}) + 2c_{12}(\lambda z)\text{Tr}(A_{12}) + c_{22}(\lambda z)\text{Tr}(A_{22})) + \lambda^2 \langle b(\lambda z), p \rangle. \end{aligned}$$

We obtain $c_{ij}(\lambda z) = c_{ij}(z)$ and $\lambda b(\lambda z) = b(z)$. We obtain that $c_{ij}(z) = c_{ij}$ are constants and $b(z)$ is homogeneous of degree -1 .

An example is given by

$$(4.25) \quad \begin{aligned} C_t &= c_{11} \Delta_x C + 2c_{12} \text{Tr}(D_{xy}^2 C) + c_{22} \Delta_y C \\ &+ \frac{\kappa_1}{|x-y|^2} \langle D_x C, x-y \rangle + \frac{\kappa_2}{|x-y|^2} \langle D_y C, x-y \rangle, \end{aligned}$$

where c_{ij}, κ_i are constants. The ellipticity of F implies that (c_{ij}) is a positive definite matrix. If $c_{11} = c_{22}$ and $\kappa_2 = -\kappa_1$, then we also satisfy the axiom of symmetry with respect to the change of order of coordinates.

Remark 13. As in the computations done before the proof of Theorem 4.3, let A, B be two $N \times N$ matrices, $C(t, x, y) = \int_{\mathbb{R}^N} g_t(z) C(0, x + Ah, y + Bh) dh$, where g_t is the Gaussian of scale t , and $C(0, x, y) = (I(x) - J(y))^2$. Then $C(t, x, y)$ satisfies the equation

$$(4.26) \quad C_t = \text{Trace}(A^t A D_x^2 C) + 2 \text{Trace}(A B^t D_{xy} C) + \text{Trace}(B^t B D_y^2 C).$$

Note that this equation corresponds to the models described in Theorem 4.2, and in particular to (1.3), when the metrics are constant in both images. This will be exploited as a numerical approximation in [10], where the construction of the metrics, which is a relevant issue, will be discussed in detail. A preliminary result illustrating the comparison measure is shown in section 6.

5. The morphological axiom. In this section we assume that T_t is a multiscale analysis satisfying the axioms, including all architectural axioms, the comparison principle, and the gray level shift invariance.

Let us recall the following axiom:

[Gray scale invariance] $T_t(f(C)) = f(T_t(C)) \quad \forall t \geq 0, \forall C \in C_b^\infty(\mathcal{M}^1 \times \mathcal{M}^2)$,

and for any strictly increasing function $f : \mathbb{R} \rightarrow \mathbb{R}$.

It is also called the morphological axiom.

The next lemma can be proved as in [1, 5] (see section 5.1 and Lemma 4.1, respectively).

LEMMA 5.1. *Assume that T_t satisfies all architectural axioms, the comparison principle, and the morphological axiom.*

$$(5.1) \quad \begin{aligned} F(\lambda A + \mu p \otimes p, \lambda p, \xi, G, \Gamma^k) &= \lambda F(A, p, \xi, G, \Gamma^k) \\ \forall A \in \text{SM}_\xi(\mathcal{N}), \forall p \in T_\xi^* \mathcal{N}, \forall \xi \in \mathcal{N}, \forall \lambda \geq 0, \mu \in \mathbb{R}. \end{aligned}$$

Let $Q_p = I - \frac{p \otimes G^{-1}p}{(G^{-1}p, p)}$, $p \in T_\xi^* \mathcal{N} \setminus \{0\}$. Then $Q_p : T_\xi \mathcal{N} \rightarrow T_\xi \mathcal{N}$ and $Q_p^t : T_\xi^* \mathcal{N} \rightarrow T_\xi^* \mathcal{N}$. As in [1, 5] (see section 5.1 and Theorem 4.2, respectively), we prove the following theorem.

THEOREM 5.2. *Let T_t be a multiscale analysis satisfying the axioms, including all architectural axioms, the comparison principle, and the morphological axiom. Then*

$$(5.2) \quad \begin{aligned} F(A, p, \xi, G, \Gamma^k) &= F(Q_p^t A Q_p, p, \xi, G, \Gamma^k) \\ \forall A \in \text{SM}_\xi(\mathcal{N}), \forall p \in T_\xi^* \mathcal{N} \setminus \{0\}, \forall \xi \in \mathcal{N}. \end{aligned}$$

A similar statement holds for \tilde{F} . Let B be such that $B^t G B = I$. In terms of \mathcal{H} we have

$$(5.3) \quad \begin{aligned} \mathcal{H}(B^t(A - \Gamma(p))B, B^t p, \xi) &= \mathcal{H}(B^t Q_p^t (A - \Gamma(p)) Q_p B, B^t p, \xi) \\ \forall A \in \text{SM}_\xi(\mathcal{N}), \forall p \in T_\xi^* \mathcal{N} \setminus \{0\}, \forall \xi \in \mathcal{N}. \end{aligned}$$

By combining the computed invariances (including that with respect to diagonal rotations) we note the following lemma.

LEMMA 5.3. *We denote here by R a diagonal rotation in the sense given above. For F we have*

$$(5.4) \quad \begin{aligned} F(S, p, \xi, G, \Gamma^k) &= F(R^t Q_p^t S Q_p R, R^t p, \xi, G, \bar{\Gamma}^k) \\ \forall S \in \text{SM}_\xi(\mathcal{N}), \forall p \in T_\xi^* \mathcal{N} \setminus \{0\}, \forall \xi \in \mathcal{N}. \end{aligned}$$

For \mathcal{H} we have

$$(5.5) \quad \mathcal{H}(B^t(S - \Gamma(p))B, B^t p, \xi) = \mathcal{H}(B^t R^t Q_p^t (S - \Gamma(p)) Q_p R B, B^t R^t p, \xi).$$

At this point we do not make precise the structure of the morphologically invariant operators, since we cannot simultaneously use the same rotation with respect to both ξ_1 and ξ_2 to extract curvatures as in [1, 5]. In section 5.1 below we give some explicit examples.

5.1. Examples. We use the notation $a \otimes b(x) = (a, x)b$, $a \in T_\xi^* \mathcal{N}$, $b, x \in T_\xi \mathcal{N}$ (or when the vectors are in any of the manifolds \mathcal{M}^i). We also define $[a \otimes b](x) = \langle a, x \rangle b$, $a, b, x \in T_\xi \mathcal{N}$, so that $[a \otimes b] = G a \otimes b$.

We have defined $Q_p = I - [e_p \otimes e_p] = I - \frac{p \otimes G^{-1}p}{(G^{-1}p, p)}$, where $e_p = \frac{G^{-1}p}{|G^{-1}p|_g}$. Then $Q_p : T_\xi \mathcal{N} \rightarrow T_\xi \mathcal{N}$ and $Q_p^t : T_\xi^* \mathcal{N} \rightarrow T_\xi^* \mathcal{N}$. Recall that we have $G Q_p^{t, g} = Q_p^t G$. Regarding the eigenvalues of a matrix, the matrix acts in the same linear space. Thus

we speak about the eigenvalues of $G^{-1}Q_p^tAQ_p = G^{-1}Q_p^tGG^{-1}AQ_p = Q_p^{t,g}G^{-1}AQ_p$, where $A : T_\xi\mathcal{N} \rightarrow T_\xi^*\mathcal{N}$.

Note that $Q_p e_p = 0$ and $Q_p e = e$ for any $e \in \langle e_p \rangle^\perp$. Let $\lambda_1, \dots, \lambda_{N-1}$ and 0 be the N real eigenvalues of $G^{-1}Q_p^tAQ_p$. Note that if $N = 2$,

$$\begin{aligned} \text{Trace}_g(Q_p^tAQ_p) &= \text{Trace}(G^{-1}Q_p^tAQ_p) = \sum_{i=1}^2 \langle Q_p^{t,g}G^{-1}AQ_p e_i, e_i \rangle \\ &= \lambda_1 = \langle G^{-1}AQ_p e_p^\perp, Q_p e_p^\perp \rangle = \langle G^{-1}Ae_p^\perp, e_p^\perp \rangle. \end{aligned}$$

We denote

$$\text{curv}_g(C) = \frac{\text{Trace}_g(Q_p^tAQ_p)}{|G^{-1}p|_g} = \frac{\lambda_1}{|G^{-1}p|_g}.$$

An example is given by functions

$$F(A, p, \xi) = \mathcal{Q}(\text{Tr}_g(Q_p^tAQ_p), p, \xi) \quad \forall A \in \text{SM}_\xi(\mathcal{N}), \forall p \in (T_\xi\mathcal{N})^*, \forall \xi \in \mathcal{N},$$

where \mathcal{Q} is a nondecreasing function of its first argument. Notice that by taking $\mu = 0$ in (5.1), we have

$$\mathcal{Q}(\lambda r, \lambda p, \xi) = \lambda \mathcal{Q}(r, p, \xi) \quad \forall \lambda, r \geq 0, \forall p \in (T_\xi\mathcal{N})^*, \forall \xi \in \mathcal{N}.$$

Thus, we can write

$$(5.6) \quad \mathcal{Q}(\text{Tr}_g(Q_p^tAQ_p), p, \xi) = |G^{-1}p|_g \mathcal{Q}\left(\text{curv}_g(C), \frac{p}{|G^{-1}p|_g}, \xi\right).$$

We can take, in particular,

$$C_t = |\nabla C|_g \text{curv}_g(C),$$

where $\nabla C = G^{-1}DC$. Other examples are

$$C_t = |\nabla C|_g \text{curv}_g(C) + \alpha \frac{\langle P(\xi) \nabla_{\xi_1} C, \nabla_{\xi_2} C \rangle}{|\nabla C|_g}$$

for $\alpha \in \mathbb{R}$.

Let us specify the operator $\text{Tr}_g(Q_p^tAQ_p)$. Let $A \in \text{SM}_\xi(\mathcal{N})$ ($A_{21} = A_{12}^t$),

$$\begin{aligned} A &= \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}, \quad p = \begin{pmatrix} p_1 \\ p_2 \end{pmatrix}, \\ \bar{Q}_{p_i} &= I - \frac{p_i \otimes (G^i)^{-1}(\xi_i)p_i}{(G^{-1}(\xi)p, p)}. \end{aligned}$$

Then

$$Q_p = \begin{pmatrix} \bar{Q}_{p_1} & -\frac{p_2 \otimes (G^1)^{-1}(\xi_1)p_1}{(G^{-1}(\xi)p, p)} \\ -\frac{p_1 \otimes (G^2)^{-1}(\xi_2)p_2}{(G^{-1}(\xi)p, p)} & \bar{Q}_{p_2} \end{pmatrix}.$$

We have

$$Q_p^tAQ_p = \begin{pmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{pmatrix}$$

with $M_{12}^t = M_{21}$. For simplicity of notation let us write $G_i^{-1} = (G^i)^{-1}(\xi_i)$, $|p|^2 := (G^{-1}(\xi)p, p) = |G^{-1}p|_g^2$,

$$\begin{aligned}
M_{11} &= \bar{Q}_{p_1}^t A_{11} \bar{Q}_{p_1} - \frac{G_2^{-1} p_2 \otimes p_1}{|p|^2} A_{21} \bar{Q}_{p_1} - \bar{Q}_{p_1}^t A_{12} \frac{p_1 \otimes G_2^{-1} p_2}{|p|^2} \\
&\quad + \frac{G_2^{-1} p_2 \otimes p_1}{|p|^2} A_{22} \frac{p_1 \otimes G_2^{-1} p_2}{|p|^2}, \\
M_{12} &= -\bar{Q}_{p_1}^t A_{11} \frac{p_2 \otimes G_1^{-1} p_1}{|p|^2} + \frac{G_2^{-1} p_2 \otimes p_1}{|p|^2} A_{21} \frac{p_2 \otimes G_1^{-1} p_1}{|p|^2} + \bar{Q}_{p_1}^t A_{12} \bar{Q}_{p_2} \\
&\quad - \frac{G_2^{-1} p_2 \otimes p_1}{|p|^2} A_{22} \bar{Q}_{p_2}, \\
M_{21} &= -\frac{G_1^{-1} p_1 \otimes p_2}{|p|^2} A_{11} \bar{Q}_{p_1} + \bar{Q}_{p_2}^t A_{21} \bar{Q}_{p_1} + \frac{G_1^{-1} p_1 \otimes p_2}{|p|^2} A_{12} \frac{p_1 \otimes G_2^{-1} p_2}{|p|^2} \\
&\quad - \bar{Q}_{p_2}^t A_{22} \frac{p_1 \otimes G_2^{-1} p_2}{|p|^2}, \\
M_{22} &= \frac{G_1^{-1} p_1 \otimes p_2}{|p|^2} A_{11} \frac{p_2 \otimes G_1^{-1} p_1}{|p|^2} - \bar{Q}_{p_2}^t A_{21} \frac{p_2 \otimes G_1^{-1} p_1}{|p|^2} - \frac{G_1^{-1} p_1 \otimes p_2}{|p|^2} A_{12} \bar{Q}_{p_2} \\
&\quad + \bar{Q}_{p_2}^t A_{22} \bar{Q}_{p_2}.
\end{aligned}$$

Observe that

$$S^t Q_p^t A Q_p S = \begin{pmatrix} M_{22} & M_{21} \\ M_{12} & M_{11} \end{pmatrix}.$$

Thus, in the case of symmetry in x, y , the function F is symmetric by interchanging M_{11} with M_{22} and M_{12} with M_{21} .

Recall that in the PDE context

$$A = D_{\mathcal{N}}^2 C = \begin{pmatrix} D_{\mathcal{M},xx} C & D_{\mathcal{M},xy} C \\ D_{\mathcal{M},yx} C & D_{\mathcal{M},yy} C \end{pmatrix}, \quad v = D_{\mathcal{N}} C = \begin{pmatrix} D_{\mathcal{M},x} C \\ D_{\mathcal{M},y} C \end{pmatrix}.$$

Example 1. The first example is $\text{Tr}_g(Q_p^t A Q_p) = \text{Tr}_g(M_{11}) + \text{Tr}_g(M_{22})$. Write $G_i = g_i = G(\xi_i)$. The traces are

$$\begin{aligned}
\text{Tr}_{g_1}(M_{11}) &= \text{Tr}_{g_1}(\bar{Q}_{p_1}^t A_{11} \bar{Q}_{p_1}) - \frac{2}{|p|^2} (A_{21} G_1^{-1} p_1, G_2^{-1} p_2) \\
&\quad + \frac{2}{|p|^4} (p_1, G_1^{-1} p_1) (A_{21} G_1^{-1} p_1, G_2^{-1} p_2) + \frac{(G_1^{-1} p_1, p_1)}{|p|^4} (A_{22} G_2^{-1} p_2, G_2^{-1} p_2), \\
\text{Tr}_{g_2}(M_{22}) &= \text{Tr}_{g_2}(\bar{Q}_{p_2}^t A_{22} \bar{Q}_{p_2}) - \frac{2}{|p|^2} (A_{12} G_2^{-1} p_2, G_1^{-1} p_1) \\
&\quad + \frac{2}{|p|^4} (p_2, G_2^{-1} p_2) (A_{12} G_2^{-1} p_2, G_1^{-1} p_1) + \frac{(G_2^{-1} p_2, p_2)}{|p|^4} (A_{11} G_1^{-1} p_1, G_1^{-1} p_1).
\end{aligned}$$

This operator satisfies all axioms, including rotation invariance. In the conformal case, the expressions can be somewhat friendly.

Example 2. We can consider all functions $\mathcal{Q}(\text{Tr}_{g_1}(M_{11}), \text{Tr}_{g_2}(M_{22}))$ which are homogeneous of degree 1 and monotone increasing and symmetric in the two variables. Examples of \mathcal{Q} are $\mathcal{Q}(a, b) = a + b$ and $\mathcal{Q}(a, b) = \sqrt{a+b}$.

Example 3. We can consider all functions $\mathcal{Q}(\text{Tr}_{g_1}(M_{11}), \text{Tr}_{g_2}(M_{22}), p)$ which are homogeneous of degree 1 (see (5.6)) in all variables and monotone increasing and

symmetric in the first two variables. We can write

$$|p|\mathcal{Q}\left(\frac{1}{|p|}\mathrm{Tr}_{g_1}(M_{11}), \frac{1}{|p|}\mathrm{Tr}_{g_2}(M_{22}), \frac{p}{|p|}\right).$$

If it does not depend on the last variable

$$|p|\mathcal{Q}\left(\frac{1}{|p|}\mathrm{Tr}_{g_1}(M_{11}), \frac{1}{|p|}\mathrm{Tr}_{g_2}(M_{22})\right) = \mathcal{Q}(\mathrm{Tr}_{g_1}(M_{11}), \mathrm{Tr}_{g_2}(M_{22})),$$

we recover Example 2. But we also have

$$\begin{aligned} |p|\mathcal{Q}\left(\frac{1}{|p|}\mathrm{Tr}_{g_1}(M_{11}), \frac{1}{|p|}\mathrm{Tr}_{g_2}(M_{22})\right) &+ \alpha \frac{\langle P(\xi)G_1^{-1}p_1, G_2^{-1}p_2 \rangle}{|p|} \\ &= \mathcal{Q}(\mathrm{Tr}_{g_1}(M_{11}), \mathrm{Tr}_{g_2}(M_{22})) + \alpha \frac{\langle P(\xi)G_1^{-1}p_1, G_2^{-1}p_2 \rangle}{|p|}, \end{aligned}$$

where $\alpha \in \mathbb{R}$. \mathcal{Q} is homogeneous of degree 1 and monotone increasing and symmetric in the two variables. Examples of \mathcal{Q} are $\mathcal{Q}(a, b) = a + b$ and $\mathcal{Q}(a, b) = \sqrt{a^+b^+}$.

Example 4. A more interesting example is

$$\mathcal{T}_g(Q_p^t A Q_p) := \mathrm{Tr}_{g_1}(M_{11}) + \mathrm{Tr}_{g_{1,P}}(M_{12}) + \mathrm{Tr}_{g_{2,P}}(M_{21}) + \mathrm{Tr}_{g_2}(M_{22}),$$

where

$$\begin{aligned} \mathrm{Tr}_{g_{1,P}}(M_{12}) &= \mathrm{Tr}(P(\xi_1, \xi_2)G_1^{-1}M_{12}), \\ \mathrm{Tr}_{g_{2,P}}(M_{21}) &= \mathrm{Tr}(P(\xi_2, \xi_1)G_2^{-1}M_{21}). \end{aligned}$$

Note that the operators $G_1^{-1}M_{12}$ and $G_2^{-1}M_{21}$ are not endomorphisms. Thus, we need the operators $P(\xi_1, \xi_2)$ and $P(\xi_2, \xi_1)$. Thus, these are mixed traces.

The remaining trace is then

$$\begin{aligned} \mathrm{Tr}_{g_{1,P}}(M_{12}) &= -\frac{1}{|p|^2}(P(\xi)G_1^{-1}A_{11}G_1^{-1}p_1, p_2) + \frac{1}{|p|^4}(P(\xi)G_1^{-1}p_1, p_2)(A_{11}G_1^{-1}p_1, G_1^{-1}p_1) \\ &+ \frac{1}{|p|^4}(P(\xi)G_1^{-1}p_1, p_2)(A_{21}G_1^{-1}p_1, G_2^{-1}p_2) + \mathrm{Tr}_{g_{1,P}}(A_{12}) - \frac{1}{|p|^2}(P(\xi)G_1^{-1}A_{12}G_2^{-1}p_2, p_2) \\ &\quad - \frac{1}{|p|^2}(P(\xi)G_1^{-1}p_1, A_{21}G_1^{-1}p_1) + \frac{1}{|p|^4}(P(\xi)G_1^{-1}p_1, p_2)(A_{12}G_2^{-1}p_2, G_1^{-1}p_1) \\ &\quad - \frac{1}{|p|^2}(P(\xi)G_1^{-1}p_1, A_{22}G_2^{-1}p_2) + \frac{1}{|p|^4}(P(\xi)G_1^{-1}p_1, p_2)(A_{22}G_2^{-1}p_2, G_2^{-1}p_2). \end{aligned}$$

The trace $\mathrm{Tr}_{g_{2,P}}(M_{21})$ has the same expression interchanging the indexes 1 and 2 and using the a priori connection between ξ_2 and ξ_1 , $P(\xi_2, \xi_1)$. Notice that by definition is elliptic. It is also rotation invariance since traces are.

We can also consider the operator

$$\mathcal{T}_g(Q_p^t A Q_p) + \alpha \frac{\langle P(\xi)G_1^{-1}p_1, G_2^{-1}p_2 \rangle}{|p|},$$

where $\alpha \in \mathbb{R}$.

Remark 14. Note that $\frac{\langle P(\xi)G_1^{-1}p_1, G_2^{-1}p_2 \rangle}{|p|}$ corresponds to $\frac{\langle P(\xi)\nabla_{\xi_1}C, \nabla_{\xi_2}C \rangle}{|\nabla C|_g}$.

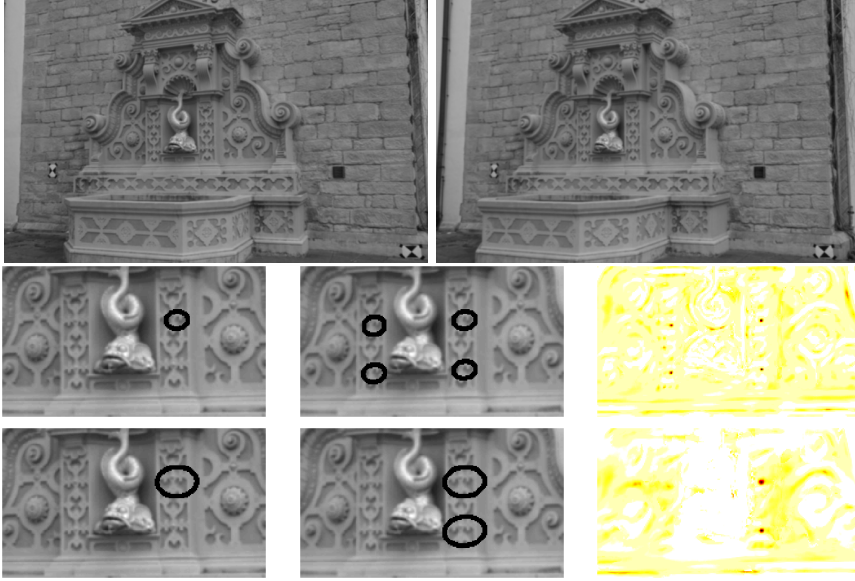


FIG. 1. Illustration of a similarity measure. The values of the similarity measure are computed between one point x in the reference image (first row, left) and all points y of the secondary image (first row, right), which is taken from a different viewpoint. In the second row we show a close-up containing the point x and the value of the similarity measure. Red pixels denote lower values of $C(t, x, y)$. Note that the minima of the similarity measure occur at the points where the structure of the secondary image is similar to the reference patch. The comparison windows are ellipses which correspond to the unit ball mapped according to A and B . The third row illustrates the similarity landscape for a larger scale. Varying the scale of the analysis corresponds in this case to increasing the window size.

Let us observe that the expression $\langle P(\xi)G_1^{-1}p_1, G_2^{-1}p_2 \rangle$ is invariant with respect to diagonally related rotations. Let $v_i \in T_{\xi_i}\mathcal{M}$. Let $R = (R_1, R_2)$ be a diagonally related rotation so that

$$(5.7) \quad R_2 = P(\xi)R_1\overline{P}(\xi)^{-1}.$$

Let us recall that given an isometry (rotation) in the tangent plane, covector gradients p_i transform as $R^t p_i$. Its associated vector is $G_i^{-1}R^t p_i = R^{t,g}G_i^{-1}p_i = R^{t,g}v_i$ where $v_i = G_i^{-1}p_i$ is the vector associated to p_i .

Since $R_i^{t,g} = R_i^{-1}$, we write (5.7) as

$$\overline{P}(\xi)R_1^{t,g} = R_2^{t,g}P(\xi).$$

Then

$$\langle \overline{P}(\xi)R_1^{t,g}v_1, R_2^{t,g}v_2 \rangle = \langle R_2^{t,g}P(\xi)v_1, R_2^{t,g}v_2 \rangle = \langle P(\xi)v_1, v_2 \rangle.$$

This is the required invariance.

Remark 15. In the conformal case, the expressions can be somewhat friendly. In the conformal case, we have

$$\mathrm{Tr}_{g_1, P}(M_{12}) = \mathrm{Tr}_{g_2, P}(M_{21}) = \frac{1}{\lambda^1(x)\lambda^2(y)}\mathrm{Tr}(M_{12}),$$

and the term $\frac{\langle P(\xi)G_1^{-1}p_1, G_2^{-1}p_2 \rangle}{|p|}$ writes as

$$\frac{1}{\lambda^1(x)\lambda^2(y)}(D_x C, D_y C).$$

Remark 16. The next example does not fall under either of the classes above. It is neither morphologically invariant nor linear. We construct it as the sum of a linear operator and a first order one that satisfies the morphological invariance

$$(5.8) \quad C_t = \frac{a(x, y)}{(\lambda^1(x))^2} \Delta_{\mathcal{M}_x} C + 2 \frac{c_{12}(x, y)}{\lambda^1(x)\lambda^2(y)} \text{Tr}(D_{xy} C) + \frac{c(x, y)}{(\lambda^2(y))^2} \Delta_{\mathcal{M}_y} C \\ + \alpha \frac{\langle P(\xi) \nabla_{\xi_1} C, \nabla_{\xi_2} C \rangle}{|\nabla C|_g},$$

where $\alpha \in \mathbb{R}$. Note that the operator is homogeneous of degree 1 in C .

6. Conclusions. In this paper we define a multiscale comparison of images defined on Riemannian manifolds. Given two images u and v , we introduce intrinsic multiscale similarity measures to compare their neighborhoods at the points $x, y \in \mathbb{R}^2$, respectively. This could be also applied to the problem of comparing two patches of an image defined on a Riemannian manifold, which can be defined on the image domain with a suitable metric depending on the image. This paper contains mostly theoretical results, some (mostly linear) examples of such measures, and the case of some morphological scale spaces. These similarity measures are useful for the purpose of computing disparities and correspondences, and determining the most similar patch, which will be the subject of a future paper [10].

We include here a preliminary result illustrating the comparison measure proposed in Remark 13. For this measure the matrices A and B are related to the prior connection. They are defined using anisotropic metrics on the images which are similar to those used in [5]. Figure 1 illustrates the values $C(t, x, y)$ of this similarity measure computed between a fixed point x in the reference image and all points y of the secondary image, which is taken from a different viewpoint. Note that the peaks of the similarity measure occur at the points where the structure of the secondary image is similar to the reference patch. We depict the comparison windows on some of the points. The comparison windows are ellipses which correspond to the unit ball mapped according to A and B . The second row illustrates the similarity landscape for a larger scale. Varying the scale of the analysis corresponds in this case to increasing the window size.

REFERENCES

- [1] L. ALVAREZ, F. GUICHARD, P.-L. LIONS, AND J.-M. MOREL, *Axioms and fundamental equations of image processing*, Arch. Ration. Mech. Anal., 123 (1993), pp. 199–257.
- [2] L. ALVAREZ, P.-L. LIONS, AND J.-M. MOREL, *Image selective smoothing and edge detection by nonlinear diffusion. II*, SIAM J. Numer. Anal., 29 (1992), pp. 845–866.
- [3] T. BROX, R. BOOMGAARD, F. LAUZE, J. WELJER, J. WEICKERT, P. MRÁZEK, AND P. KORNPROBST, *Adaptive structure tensors and their applications*, in Visualization and Processing of Tensor Fields, Springer, Berlin, 2006, pp. 17–47.
- [4] T. BROX, J. WEICKERT, B. BURGETH, AND P. MRÁZEK, *Nonlinear structure tensors*, Image Vis. Comput., 24 (2006), pp. 41–55.
- [5] F. CALDERERO AND V. CASELLES, *Multiscale analysis of images on Riemannian manifolds. Part I*, submitted.
- [6] F. CAO, J.L. LISANI, J.-M. MOREL, P. MUSÉ, AND F. SUR, *A Theory of Shape Identification*, Lecture Notes in Math. 1948, Springer, Berlin, 2008.

- [7] V. CASELLES, L. IGUAL, AND O. SANDER, *An axiomatic approach to scalar data interpolation on surfaces*, Numer. Math., 102 (2006), pp. 383–411.
- [8] V. CASELLES, J.-M. MOREL, AND C. SBERT, *An axiomatic approach to image interpolation*, IEEE Trans. Image Process., 7 (1998), pp. 376–386.
- [9] F. CATTÉ, P.-L. LIONS, J.-M. MOREL, AND T. COLL, *Image selective smoothing and edge detection by nonlinear diffusion*, SIAM J. Numer. Anal., 29 (1992), pp. 182–193.
- [10] G. FACCILOLO, F. CALDERERO, C. BALLESTER, AND V. CASELLES, *Multiscale analysis of similarities between images on Riemannian manifolds. Part II*, in preparation, 2013.
- [11] Y. GIGA AND S. GOTO, *Motion of hypersurfaces and geometric equations*, J. Math. Soc. Japan, 44 (1992), pp. 99–111.
- [12] Y. GIGA, S. GOTO, H. ISHII, AND M.H. SATO, *Comparison principle and convexity preserving properties for singular degenerate parabolic equations on unbounded domains*, Indiana Univ. Math. J., 40 (1991), pp. 443–470.
- [13] M. GRAYSON, *The heat equation shrinks embedded plane curves to round points*, J. Differential Geom., 26 (1987), pp. 285–314.
- [14] F. GUICHARD, *Axiomatization of Images and Movies Scalespace*, Ph.D. thesis, University of Paris IX-Dauphine, Paris, France, 1994.
- [15] F. GUICHARD, *A morphological, affine, and Galilean invariant scale-space for movies*, IEEE Trans. Image Process., 7 (1998), pp. 444–456.
- [16] F. GUICHARD AND J.M. MOREL, *Image analysis and PDEs*, IPAM GBM Tutorial, IPAM, UCLA, Los Angeles, CA, 2001.
- [17] R.A. HUMMEL, *The scale-space formulation of pyramid data structures*, in Parallel Computer Vision, Academic Press Professional, New York, 1987, pp. 107–123.
- [18] R.A. HUMMEL, B.B. KIMIA, AND S.W. ZUCKER, *Gaussian blur and the heat equation: Forward and inverse solutions*, in Proceedings of the IEEE International Conference on Computer Vision and Pattern Recognition, IEEE, Washington, DC, 1985, pp. 668–671.
- [19] K. KANATANI, *Group-Theoretical Methods in Image Understanding, Vol. 2*, Springer-Verlag, Berlin, 1990.
- [20] B.B. KIMIA, A. TANNENBAUM, AND S. ZUCKER, *Toward a computational theory of shape: An overview*, in Computer Vision: ECCV 90, Lecture Notes in Comput. Sci. 427, Springer-Verlag, Berlin, 1990, pp. 402–407.
- [21] B.B. KIMIA, A. TANNENBAUM, AND S.W. ZUCKER, *On the evolution of curves via a function of curvature. I. The classical case*, J. Math. Anal. Appl., 163 (1992), pp. 438–458.
- [22] R. KIMMEL, R. MALLADI, AND N. SOCHEN, *Images as embedded maps and minimal surfaces: Movies, color, texture, and volumetric medical images*, Internat. J. Comput. Vis., 39 (2000), pp. 111–129.
- [23] R. KIMMEL, N. SOCHEN, AND R. MALLADI, *From high energy physics to low level vision*, in Scale-Space Theory in Computer Vision, Springer-Verlag, Berlin, 1997, pp. 236–247.
- [24] J.J. KOENDERINK, *The structure of images*, Biol. Cybernet., 50 (1984), pp. 363–370.
- [25] J.J. KOENDERINK AND A.J. VAN DOORN, *Dynamic shape*, Biol. Cybernet., 53 (1986), pp. 383–396.
- [26] T. LINDBERG, *Scale-Space Theory in Computer Vision*, Kluwer Academic Publishers, Dordrecht, The Netherlands, 1993.
- [27] D.G. LOWE, *Distinctive image features from scale-invariant keypoints*, Internat. J. Comput. Vis., 60 (2004), pp. 91–110.
- [28] D. MARR AND E. HILDRETH, *Theory of edge detection*, Proc. Roy. Soc. London Ser. B Biol. Sci., 207 (1980), pp. 187–217.
- [29] S. MASNOU, *Disocclusion: A variational approach using level lines*, IEEE Trans. Image Process., 11 (2002), pp. 68–76.
- [30] S. MASNOU AND J.-M. MOREL, *Level lines based disocclusion*, in Proceedings of the IEEE International Conference on Image Processing, IEEE, Washington, DC, 1998, pp. 259–263.
- [31] F. MOKHTARIAN AND A. MACKWORTH, *Scale-based description and recognition of planar curves and two-dimensional shapes*, IEEE Trans. Pattern Anal. Machine Intell., 8 (1986), pp. 34–43.
- [32] P.J. OLVER, G. SAPIRO, AND A. TANNENBAUM, *Differential Invariant Signatures and Flows in Computer Vision: A Symmetry Group Approach*, Technical report, DTIC Document, 1993.
- [33] S. OSHER AND J.A. SETHIAN, *Fronts propagating with curvature-dependent speed: Algorithms based on Hamilton-Jacobi formulations*, J. Comput. Phys., 79 (1988), pp. 12–49.
- [34] P. PERONA AND J. MALIK, *Scale-space and edge detection using anisotropic diffusion*, IEEE Trans. Pattern Anal. Machine Intell., 12 (1990), pp. 629–639.
- [35] G. PEYRÉ, *Manifold models for signals and images*, Comput. Vis. Image Understanding, 113

- (2009), pp. 249–260.
- [36] L.I. RUDIN, S. OSHER, AND E. FATEMI, *Nonlinear total variation based noise removal algorithms*, Phys. D, 60 (1992), pp. 259–268.
 - [37] G. SAPIRO AND A. TANNENBAUM, *Affine invariant scale-space*, Internat. J. Comput. Vis., 11 (1993), pp. 25–44.
 - [38] G. SAPIRO AND A. TANNENBAUM, *On affine plane curve evolution*, J. Funct. Anal., 119 (1994), pp. 79–120.
 - [39] J. SERRA, *Image Analysis and Mathematical Morphology*, Academic Press, New York, 1982.
 - [40] N. SOCHEN, R. KIMMEL, AND A.M. BRUCKSTEIN, *Diffusions and confusions in signal and image processing*, J. Math. Imaging Vis., 14 (2001), pp. 195–209.
 - [41] N. SOCHEN, R. KIMMEL, AND R. MALLADI, *A general framework for low level vision*, IEEE Trans. Image Process., 7 (1998), pp. 310–318.
 - [42] E. STIEFEL, *Richtungsfelder und fernparallelismus in n -dimensionalen mannigfaltigkeiten*, Comment. Math. Helv., 8 (1935), pp. 305–353.
 - [43] P.J. OLVER, G. SAPIRO, AND A. TANNENBAUM, *Invariant geometric evolutions of surfaces and volumetric smoothing*, SIAM J. Appl. Math., 57 (1997), pp. 176–194.
 - [44] E. THOMAS, *Vector fields on manifolds*, Bull. Amer. Math. Soc., 75 (1969), pp. 643–683.
 - [45] J. WEICKERT, *Theoretical foundations of anisotropic diffusion in image processing*, Computing Wien Supplements, 11 (1996), pp. 221–236.
 - [46] J. WEICKERT, *Anisotropic Diffusion in Image Processing, Vol. 1*, Teubner, Stuttgart, Germany, 1998.
 - [47] J. WEICKERT, *Coherence-enhancing diffusion filtering*, Internat. J. Comput. Vis., 31 (1999), pp. 111–127.
 - [48] J. WEICKERT, S. ISHIKAWA, AND A. IMIYA, *Linear scale-space has first been proposed in Japan*, J. Math. Imaging Vis., 10 (1999), pp. 237–252.
 - [49] A. WITKIN, *Scale-space filtering: A new approach to multi-scale description*, in Proceedings of the IEEE International Conference on Acoustics, Speech, and Signal Processing, Vol. 9, IEEE, Washington, DC, 1984, pp. 150–153.